Finally, we end this section with a discussion of analytic continuation (the third of the "miracles" we mentioned in the introduction). It states that the "genetic code" of a holomorphic function is determined (that is, the function is fixed) if we know its values on appropriate arbitrarily small subsets. Note that in the theorem below, $\Omega$ is assumed connected.

**Theorem 4.8** *Suppose $f$ is a holomorphic function in a region $\Omega$ that vanishes on a sequence of distinct points with a limit point in $\Omega$. Then $f$ is identically* 0.

In other words, if the zeros of a holomorphic function $f$ in the connected open set $\Omega$ accumulate *in* $\Omega$, then $f = 0$.

*Proof.* Suppose that $z_0 \in \Omega$ is a limit point for the sequence $\{w_k\}_{k=1}^{\infty}$ and that $f(w_k) = 0$. First, we show that $f$ is identically zero in a small disc containing $z_0$. For that, we choose a disc $D$ centered at $z_0$ and contained in $\Omega$, and consider the power series expansion of $f$ in that disc

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

If $f$ is not identically zero, there exists a smallest integer $m$ such that $a_m \neq 0$. But then we can write

$$f(z) = a_m (z - z_0)^m (1 + g(z - z_0)),$$

where $g(z - z_0)$ converges to 0 as $z \to z_0$. Taking $z = w_k \neq z_0$ for a sequence of points converging to $z_0$, we get a contradiction since $a_m(w_k - z_0)^m \neq 0$ and $1 + g(w_k - z_0) \neq 0$, but $f(w_k) = 0$.

We conclude the proof using the fact that $\Omega$ is connected. Let $U$ denote the interior of the set of points where $f(z) = 0$. Then $U$ is open by definition and non-empty by the argument just given. The set $U$ is also closed since if $z_n \in U$ and $z_n \to z$, then $f(z) = 0$ by continuity, and $f$ vanishes in a neighborhood of $z$ by the argument above. Hence $z \in U$. Now if we let $V$ denote the complement of $U$ in $\Omega$, we conclude that $U$ and $V$ are both open, disjoint, and

$$\Omega = U \cup V.$$

Since $\Omega$ is connected we conclude that either $U$ or $V$ is empty. (Here we use one of the two equivalent definitions of connectedness discussed in Chapter 1.) Since $z_0 \in U$, we find that $U = \Omega$ and the proof is complete.

An immediate consequence of the theorem is the following.

**Corollary 4.9** *Suppose $f$ and $g$ are holomorphic in a region $\Omega$ and $f(z) = g(z)$ for all $z$ in some non-empty open subset of $\Omega$ (or more generally for $z$ in some sequence of distinct points with limit point in $\Omega$). Then $f(z) = g(z)$ throughout $\Omega$.*

Suppose we are given a pair of functions $f$ and $F$ analytic in regions $\Omega$ and $\Omega'$, respectively, with $\Omega \subset \Omega'$. If the two functions agree on the smaller set $\Omega$, we say that $F$ is an **analytic continuation** of $f$ into the region $\Omega'$. The corollary then guarantees that there can be only one such analytic continuation, since $F$ is uniquely determined by $f$.