EXAMPLE 2. An integral that will play an important role in Chapter 6 (Gamma function) is

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{\pi}{\sin \pi a}, \qquad 0 < a < 1.$$

To prove this formula, let $f(z) = e^{az}/(1+e^z)$, and consider the contour consisting of a rectangle in the upper half-plane with a side lying

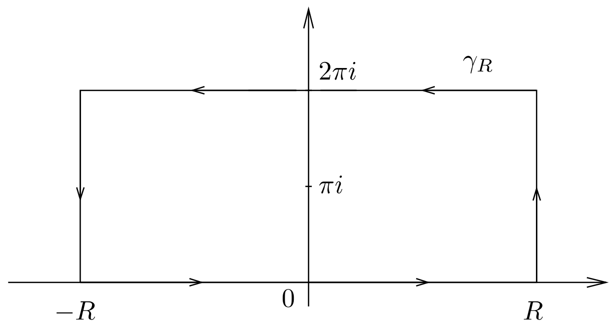

Figure 2. The contour γ_R in Example 2

on the real axis, and a parallel side on the line $\mathrm{Im}(z) = 2\pi$, as shown in Figure 2.

The only point in the rectangle γ_R where the denominator of f vanishes is $z = \pi i$. To compute the residue of f at that point, we argue as follows: First, note

$$(z - \pi i)f(z) = e^{az}\frac{z - \pi i}{1 + e^z} = e^{az}\frac{z - \pi i}{e^z - e^{\pi i}}.$$

We recognize on the right the inverse of a difference quotient, and in fact

$$\lim_{z \to \pi i} \frac{e^z - e^{\pi i}}{z - \pi i} = e^{\pi i} = -1$$

since e^z is its own derivative. Therefore, the function f has a simple pole at πi with residue

$$\mathrm{res}_{\pi i} f = -e^{a\pi i}.$$

As a consequence, the residue formula says that

$$\int_{\gamma_R} f = -2\pi i e^{a\pi i}. \tag{3}$$

We now investigate the integrals of f over each side of the rectangle. Let I_R denote

$$\int_{-R}^{R} f(x)\,dx$$

and I the integral we wish to compute, so that $I_R \to I$ as $R \to \infty$. Then, it is clear that the integral of f over the top side of the rectangle (with the orientation from right to left) is

$$-e^{2\pi i a} I_R.$$

Finally, if $A_R = \{R + it : \ 0 \le t \le 2\pi\}$ denotes the vertical side on the right, then

$$\left| \int_{A_R} f \right| \le \int_0^{2\pi} \left| \frac{e^{a(R+it)}}{1 + e^{R+it}} \right| dt \le Ce^{(a-1)R},$$

and since $a < 1$, this integral tends to 0 as $R \to \infty$. Similarly, the integral over the vertical segment on the left goes to 0, since it can be bounded by Ce^{-aR} and $a > 0$. Therefore, in the limit as R tends to infinity, the identity (3) yields

$$I - e^{2\pi i a} I = -2\pi i e^{a\pi i},$$

from which we deduce

$$\begin{aligned} I &= -2\pi i \frac{e^{a\pi i}}{1 - e^{2\pi i a}} \\ &= \frac{2\pi i}{e^{\pi i a} - e^{-\pi i a}} \\ &= \frac{\pi}{\sin \pi a}, \end{aligned}$$

and the computation is complete.

EXAMPLE 3. Now we calculate another Fourier transform, namely

Theorem 1.4 *For all $s \in \mathbb{C}$,*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}. \tag{4}$$

Observe that $\Gamma(1-s)$ has simple poles at the positive integers $s = 1, 2, 3, \ldots$, so that $\Gamma(s)\Gamma(1-s)$ is a meromorphic function on $\mathbb{C}$ with simple poles at *all* the integers, a property also shared by $\pi/\sin \pi s$.

To prove the identity, it suffices to do so for $0 < s < 1$ since it then holds on all of $\mathbb{C}$ by analytic continuation.

Lemma 1.5 *For $0 < a < 1$, $\displaystyle\int_0^\infty \frac{v^{a-1}}{1+v}\,dv = \frac{\pi}{\sin \pi a}$.*

Proof. We observe first that

$$\int_0^\infty \frac{v^{a-1}}{1+v}\,dv = \int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}\,dx\,,$$

which follows by making the change of variables $v = e^x$. However, using contour integration, we saw in Example 2 of Section 2.1 in Chapter 3, that the second integral equals $\pi/\sin \pi a$, as desired.

To establish the theorem, we first note that for $0 < s < 1$ we may write

$$\Gamma(1-s) = \int_0^\infty e^{-u}u^{-s}\,du = t\int_0^\infty e^{-vt}(vt)^{-s}\,dv\,,$$

where for $t > 0$ we made the change of variables $vt = u$. This trick then gives

$$\begin{aligned}
\Gamma(1-s)\Gamma(s) &= \int_0^\infty e^{-t}t^{s-1}\Gamma(1-s)\,dt \\
&= \int_0^\infty e^{-t}t^{s-1}\left(t\int_0^\infty e^{-vt}(vt)^{-s}dv\right)\,dt \\
&= \int_0^\infty\int_0^\infty e^{-t[1+v]}v^{-s}\,dvdt \\
&= \int_0^\infty \frac{v^{-s}}{1+v}\,dv \\
&= \frac{\pi}{\sin \pi(1-s)} \\
&= \frac{\pi}{\sin \pi s},
\end{aligned}$$

and the theorem is proved.