# Math 113 Homework 9

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There are four problems due Wednesday, April 17.

1. Give examples of subrings $R$ of $\mathbb{C}$ satisfying the following containments and non-containments. If it's not possible, explain why not:
   (a) $\mathbb{Q} \subset R$, $R \subset \mathbb{R}$ ($\mathbb{Q} \neq R \neq \mathbb{R}$).
   (b) $\mathbb{Z} \subset R$, $\mathbb{Q} \not\subset R$ ($R \neq \mathbb{Z}$.
   (c) $\mathbb{R} \subseteq R$, $R \subseteq \mathbb{C}$ ($\mathbb{R} \neq R \neq \mathbb{C}$).

2. Which of the following sets are ideals in the given ring?
   (a) $\{p(x,y) \mid p(x,x) = 0\} \subseteq \mathbb{C}[x,y]$
   (b) $\{p(x,y) \mid p(x,y) = p(y,x)\} \subseteq \mathbb{C}[x,y]$
   (c) $\{p(x) \mid p \text{ has no real roots}\} \subseteq \mathbb{C}[x]$

3. Let $R$ be a commutative ring. Recall that there is a unique homomorphism from $\mathbb{Z}$ to $R$. For two rings $A$ and $B$, let $\mathrm{Hom}(A,B)$ denote the set of ring homomorphisms from $A$ to $B$.
   (a) Give an example of $R$ for which $\mathrm{Hom}(R,\mathbb{Z})$ is empty.
   (b) Give an example of $R$ for which $\mathrm{Hom}(R,\mathbb{Z})$ is infinite.
   (c) Prove that the set $\mathrm{Hom}(\mathbb{Z}[x],R)$ can be naturally put into bijection with the set $R$.

4. Let $R$ be a commutative ring with unity.
   (a) Let $X \subseteq R$ be an arbitrary subset. Prove that there exists an ideal $I \subseteq R$ containing $X$ with the following property: if $J$ is an ideal and $X \subseteq J$, then $I \subseteq J$. (We call $I$ the *ideal generated by* $X$, and denote it $(X) \subseteq R$.)
   (b) If $m,n \in \mathbb{Z}$, when is the ideal generated by $\{m,n\}$ equal to all of $\mathbb{Z}$?
   (c) Determine $(X)$ when $X = \{x-1, x+1\}$ and $R = \mathbb{R}[x]$.