Math 113 Homework 10

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There are six problems due Wednesday, April 24.

1. Is there an integral domain containing exactly 10 elements?
2. Let R be an integral domain of characteristic p. Consider the map $\phi\colon R \to R$ sending x to x^p.
   (a) Show that ϕ is a ring homomorphism.
   (b) Show that ϕ is an automorphism if R is finite.
   (c) Find the image of ϕ when $R = \mathbb{Z}/p\mathbb{Z}[x]$.
3. Show that $\mathbb{Q}[\sqrt{2}, \sqrt{3}] = \mathbb{Q}[\sqrt{2}+\sqrt{3}]$. [Hint: show this by showing that if T is a subring of $\mathbb{C}$ containing $\mathbb{Q}$, then T contains $\sqrt{2}$ and $\sqrt{3}$ iff it contains $\sqrt{2}+\sqrt{3}$.]
4. Show that the ring $\mathbb{Z}[\sqrt{2}]$ has infinitely many units.
5. Find all $x \in \mathbb{Z}/16\mathbb{Z}$ such that $x^2 = 1$.
6. Give examples of the following:
   (a) A ring R where 1_R has infinite additive order, and R has zero divisors.
   (b) An ideal $I \subseteq \mathbb{C}[X]$ for which there exists $f(X) \in \mathbb{C}[X]$ such that $f(X)^5 \in I$, but $f(X) \notin I$.