Math 113 Homework 1

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Problems 1-7 should be written up (on paper or LaTeX) and handed in via Gradescope on Wednesday, February 6.

The words "map" and "function" mean the same thing.

1. Let S be a set with six elements, and let T be a set with five elements.
   (a) Find the number of functions from S to T.
   (b) Find the number of bijections from S to itself.
2. Let A, B, and C be sets, let $f\colon A \to B$ be a function from A to B, and let $g, h\colon B \to C$ be functions from B to C. Suppose that
   $$g \circ f = h \circ f.$$
   (a) Show that if f is surjective, then $g = h$ (i.e., show that every element of B is sent to the same thing under g and h!)
   (b) Find a counterexample, where $g \neq h$, but f is not surjective.
3. Show that $f\colon A \to B$ is injective if and only if it has a *left inverse*, i.e., a map $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$.
4. Show that if $f\colon A \to B$ is bijective, then there is a map $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$, and $f \circ g = \mathrm{id}_B$. [Hint: use problems 2a and 3 together.]
5. (a) Find a set S, and a bijection between S and a proper subset of S.
   (b) What condition on S guarantees that S cannot be in bijection with a proper subset of itself? [Hint: think about the sets in Problem 1]
6. Consider the relation on $\mathbb{R}$ for which $x \sim y$ if and only if $x - y$ is an integer. Show that this is an equivalence relation.

7. Let $S = \{1, 2, 3, 4, 5, 6\}$ and $S_1 = \{1, 2, 3, 4\}$ and $S_2 = \{3, 5\}$. Define the relation for which $m \sim n$ if and only if m and n are both in S_1 or both in S_2 (remember, or is not the same as xor, so it's fine if they are both in both). Does this give an equivalence relation? If not, which of the properties of an equivalence relation does it satisfy?

Optional Problems

Look at the problems below and make sure you understand how you might do them. You don't need to write them up. But it gives you an idea of the kind of concepts you need to understand in this course. You can also use them later on as exam practice if you like.

8. Let C be the set of numbers appearing on a standard clock face. We define a relation $U \subseteq C \times C$ such that $(a, b) \in U$ if $a = b$, or a is next to b on the clock. Is U an equivalence relation? If not, which properties does it satisfy?

9. (a) Let $f\colon S \to S'$ be a map of sets. Define a relation $U \subseteq S \times S$ where $(x, y) \in U$ iff $f(x) = f(y)$. Show that U is an equivalence relation.
   (b) In the case of $f\colon \mathbb{R}^2 \to \mathbb{R}$ where $f(x, y) = x - y$, describe the equivalence classes geometrically.

10. Let S be a set and $U \subseteq S \times S$ an equivalence relation on S. If $a, b \in S$, and $b \in [a]$, prove that $[b] = [a]$.

11. In the notation of the previous problem that if $[a]$ and $[b]$ have a nonempty intersection, then $[a] = [b]$ (as subsets of S).

12. Let S and T be two sets and $f\colon S \to T$ a function. Under what condition is it true that f is injective if and only if it's surjective?

13. Let S and T be sets. If $U \subseteq S$ and $V \subseteq T$, then $U \times V \subseteq S \times T$. Are all subsets of $S \times T$ of this form? If yes, prove it, but if not, find a counterexample.

14. Consider the function $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^3 + 3x$. Is it injective? Is it surjective? Justify your answer. [Hint: think about its derivative.]

15. Let S be a set with five elements.
   (a) Find the number of functions from S to itself.
   (b) Find the number of bijections from S to itself.