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Universality of Nash Equilibria

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Every shape arises in game theory

Game Theory Intro

Bertrand Russell's Game of "Chicken"

On a deserted road, two cars race towards each other

He who swerves is "chicken"

Model as a game:

Jim's Payoff (J_{lm})

		Buzz	
			
Jim		-1	2
		0	1

Buzz's Payoff (B_{lm})

		Buzz	
			
Jim		-1	0
		2	1

2 players, each with 2 pure strategies

1) don't swerve    2) swerve 

Nash Equilibrium

A choice of strategies by all players is a Nash equilibrium iff for each player, if all of the other players' strategies are held fixed, that player cannot increase his own payoff by changing only his own strategy.

In "Chicken", there are two pure-strategy Nash equilibria:

(Jim doesn't swerve *, Buzz swerves* *)*
(Jim swerves *, Buzz doesn't swerve* *)*

Mixed Strategies

A player X may move by rolling a die. If X has $d_X + 1$ *pure strategies, a* ***mixed strategy*** *of X is a* $(d_X + 1)$*-tuple,* $(x_0, x_1, \ldots, x_{d_X})$*, where* x_i *is the probability that X chooses X's ith pure strategy. So* $0 \leq x_i \leq 1$*, and* $\sum_{i=0}^{d_X} x_i = 1$*. The mixed strategies of X are points of the probability simplex.*

Payoff Functions

A choice of (mixed) strategies by all players is a strategy profile. For any player X, the payoff function for a strategy profile is the expected value of X's payoff.

$$
\begin{aligned}
\pi_{Jim} &= \sum_{l=0}^{1} \sum_{m=0}^{1} J_{lm} j_l b_m \\
&= j_0((-1)b_0 + (2)b_1) \\
&\quad + j_1((0)b_0 + (1)b_1) \\
&= (1 - j_1)((-1)(1 - b_1) + 2b_1) \\
&\quad + j_1 b_1 \\
&= (1 - j_1)(3b_1 - 1) + j_1 b_1 \\
&= -1 + j_1 + 3b_1 - 2j_1 b_1
\end{aligned}
$$

Totally Mixed Nash Equilibria

A Nash equilibrium is *totally mixed* if every pure strategy of every player occurs with positive probability.

Lemma A strategy profile is a totally mixed Nash equilibrium iff for each player X, the payoffs π_X^i of each pure strategy i of X are all equal.

$3b_1 - 1 = b_1$, so $b_1 = 1/2$ and similarly $j_1 = 1/2$

Universality of Nash Equilibria

Theorem (RSD, 2002) Every real algebraic variety is stably equivalent to the set of totally mixed Nash equilibria of a three-person game, or an N-person game in which each player has two strategies.

Multilinear Equations

Consider a game with three players: Alice, Bob, and Critter. Equating Critter's payoffs gives $d_{Critter}$ equations of the form

$$\bullet\ a_0 b_0 + \bullet a_1 b_0 + \cdots + \bullet a_{d_{Alice}} b_0$$
$$+ \bullet a_0 b_1 + \cdots + \bullet a_{d_{Alice}} b_{d_{Bob}} = 0$$

Since $\sum_{i=0}^{d_{Alice}} a_i = \sum_{j=0}^{d_{Bob}} b_j = 1$, we can eliminate a_0 and b_0 to get

$$\bullet\ a_1 b_1 + \bullet a_1 b_2 + \cdots + \bullet a_{d_{Alice}} b_{d_{Bob}}$$
$$+ \bullet a_1 + \cdots + \bullet a_{d_{Alice}}$$
$$+ \bullet b_1 + \cdots + \bullet b_{d_{Bob}} + \bullet = 0$$

Lemma We can get any such system by choosing appropriate payoffs.

Degrees of Freedom

For each choice of pure strategies by all the other players, translating Alice's payoffs for all of her own strategies by the same constant makes no difference to the result. So for each of the $(d_{Bob} + 1) \times (d_{Critter} + 1)$ *choices of pure strategies by Bob and Critter, we can set one of Alice's payoffs arbitrarily, and set the rest relative to that one. Similarly, we can set* $(d_{Alice} + 1) \times (d_{Critter} + 1)$ *of Bob's payoffs and* $(d_{Alice} + 1) \times (d_{Bob} + 1)$ *of Critter's payoffs.*

Example: Choosing payoffs

Suppose each player has 3 pure strategies, and we want to obtain the equation

$$-a_1 b_2 - b_1 + 7/64 = 0$$

We set the payoff for any player who picks their own 0*th pure strategy to* 0 *(regardless of what the other players do).*

Example Payoff

Now Critter's payoff function $\pi^i_{Critter}$ is

$$
\begin{aligned}
& C_{00i}(1-a_1-a_2)(1-b_1-b_2) \\
+ & C_{01i}(1-a_1-a_2)b_1 \\
+ & C_{02i}(1-a_1-a_2)b_2 \\
+ & C_{10i}a_1(1-b_1-b_2) \\
+ & C_{11i}a_1b_1 \\
+ & C_{12i}a_1b_2 \\
+ & C_{20i}a_2(1-b_1-b_2) \\
+ & C_{21i}a_2b_1 \\
+ & C_{22i}a_2b_2
\end{aligned}
$$

Critter's Matrix

Start at the upper left corner, then run along the top and left edges, and finally through the rest of the matrix...

$$\begin{pmatrix} \frac{7}{64} & \frac{-57}{64} & \frac{7}{64} \\ \frac{7}{64} & \frac{-57}{64} & \frac{-57}{64} \\ \frac{7}{64} & \frac{-57}{64} & \frac{7}{64} \end{pmatrix}$$

Arbitrary Polynomial Equations

Given an arbitrary system of polynomial equations, all of whose real solutions lie in the interior of the unit simplex, we want to express it as a system of multilinear equations of the above form with more variables. Projecting the solution set of this system, which is the set of totally mixed Nash equilibria of some game, onto the original variables gives us back our original solution set.

Encoding A Circle

Suppose we start with a circle,

$$a_1^2 + a_2^2 = 1.$$

We shift it and scale it so it lies entirely in the interior of the unit simplex:

$$\left(a_1 - \frac{1}{4}\right)^2 + \left(a_2 - \frac{1}{4}\right)^2 = \left(\frac{1}{8}\right)^2,$$

or

$$a_1^2 - \frac{1}{2}a_1 + a_2^2 - \frac{1}{2}a_2 + \frac{7}{64} = 0.$$

Horner's Rule

$$f_n x^n + f_{n-1} x^{n-1} + \cdots + f_0$$
$$= (\cdots(((f_n x + f_{n-1})x + f_{n-2})x + \cdots f_1)x + f_0$$

Using this to rewrite

$$a_1^2 - \frac{1}{2}a_1 + a_2^2 - \frac{1}{2}a_2 + \frac{7}{64} = 0,$$

we get

$$a_1\left(a_1 - \frac{1}{2}\right) + a_2\left(a_2 - \frac{1}{2}\right) + \frac{7}{64} = 0.$$

Payoffs For A Circle

Choose payoff matrices to obtain the following equations:

$$(A_{1\bullet\bullet}) \qquad b_1 = c_1$$

$$(A_{2\bullet\bullet}) \qquad 0 = 0$$

$$(B_{\bullet 1\bullet}) \qquad -c_2 = a_2 - \frac{1}{2}$$

$$(B_{\bullet 2\bullet}) \qquad c_1 = a_2 c_2$$

$$(C_{\bullet\bullet 1}) \qquad -b_2 = a_1 - \frac{1}{2}$$

$$(C_{\bullet\bullet 2}) \qquad 0 = -a_1 b_2 - b_1 + \frac{7}{64}$$

Algebraic Sets

An algebraic set is the set of solutions to a system of polynomial equations. It lies in a vector space over some field. The study of algebraic sets is the subject of algebraic geometry. It is convenient if the field is algebraically closed. So the most well-developed branch of algebraic geometry is complex algebraic geometry.

Real Algebraic Geometry

Real algebraic geometry is considerably different from complex algebraic geometry. For example, an irreducible complex algebraic set is connected, and unbounded unless it is a point. But the circle is a bounded real algebraic set, and the solution set of $x^2 + y^2 - x^3 = 0$ has an isolated point. A very important property of the real field is that it is *ordered*. Furthermore, the square of a real number is nonnegative. In particular, any real algebraic set is the solution set of a *single* polynomial equation:

$$\{x \mid f_1(x) = 0, \dots, f_m(x) = 0\}$$

is the same set as

$$\{x \mid f_1^2(x) + \cdots + f_m^2(x) = 0\}$$

Semialgebraic Sets

A semialgebraic set is the solution set of a system of polynomial equations and inequalities over the real numbers. The set of totally mixed Nash equilibria of a game is a semialgebraic set.

A semialgebraic mapping is a mapping $\varphi\colon X \to Y$ whose graph

$$\{(x, \varphi(x)) \mid x \in X\}$$

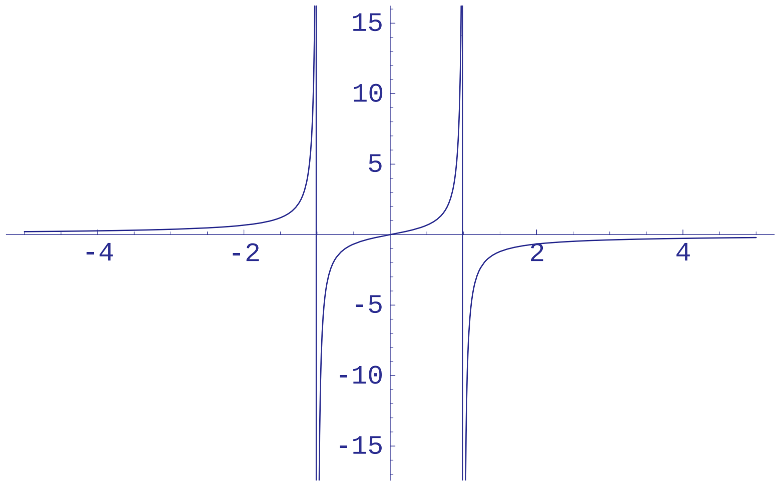
is a semialgebraic set. (We do not require the map to be polynomial or rational as in other branches of algebraic geometry, since it is convenient to allow maps involving square roots.) A semialgebraic isomorphism is a homeomorphism which is a semialgebraic mapping.

A Line Is Isomorphic To An Interval

The map

$$t \mapsto \frac{t}{1-t^2}$$

is a semialgebraic isomorphism between the open interval $(-1, 1)$ *and* $\mathbb{R}$.



Stable Equivalence

Stable equivalence is the equivalence relation generated by
(i) semialgebraic isomorphism; and
(ii) $X \cong X \times \mathbb{R}$.

A semialgebraic set is stably equivalent to the Cartesian product of itself with $\mathbb{R}^k$. The Cartesian product of a manifold with $\mathbb{R}^k$ is an example of a *tubular neighborhood* of a manifold.

Universality of Nash Equilibria

Theorem (RSD, 2002) *Every real algebraic set is stably equivalent to the set of totally mixed Nash equilibria of a three-person game, or an N-person game in which each player has two strategies.*

A Beautiful Theorem

Theorem (Nash, 1952; Tognoli, 1973) Every compact differentiable manifold is diffeomorphic to some (nonsingular) real algebraic variety.

Corollary

Every compact differentiable manifold has a tubular neighborhood which is diffeomorphic to the set of totally mixed Nash equilibria of some game.