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*Using Computer Algebra
To Compute Nash Equilibria*

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Running Example

Players: Alice, Bob, Chris

Two pure strategies each:

0 and 1

$a = \Pr[\text{Alice chooses 1}]$

$b = \Pr[\text{Bob chooses 1}]$

$c = \Pr[\text{Chris chooses 1}]$

If Alice chooses 1,

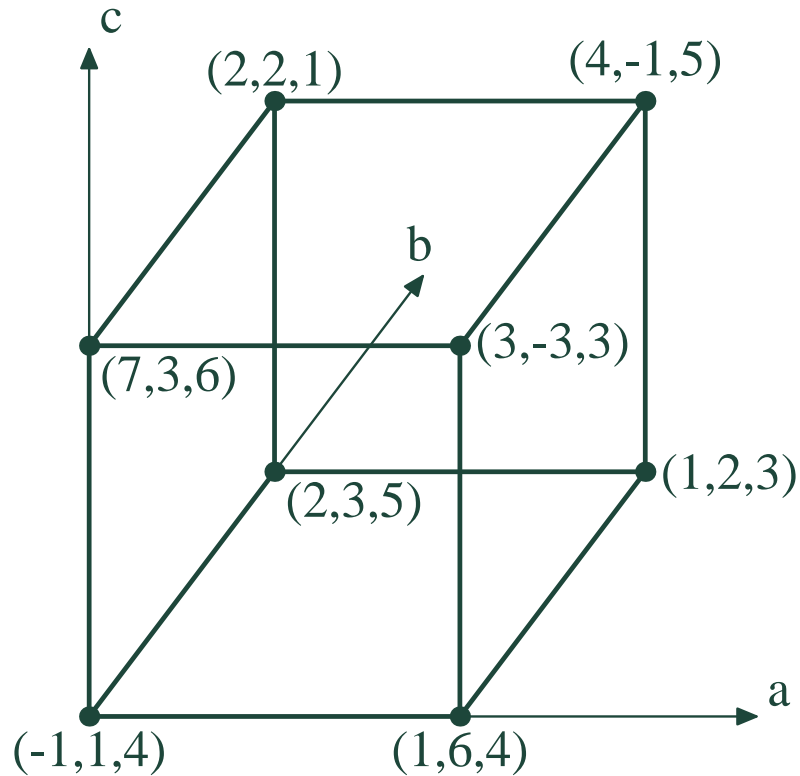
Bob chooses 0,

and Chris chooses 1,

then Alice's payoff is 3,

Bob's payoff is -3 ,

and Chris's payoff is 3.



Expected Payoff Functions

Payoff to Alice of picking pure strategy 0, is *expected payoff conditioned* on event *Alice chooses 0*:

$$2bc + 2b(1 - c) + 7(1 - b)c + (-1)(1 - b)(1 - c) = -1 + 3b + 8c - 8bc$$

Similarly, payoff to Alice from picking pure strategy 1:

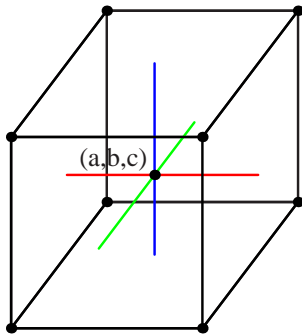
$$4bc + b(1 - c) + 3(1 - b)c + (1 - b)(1 - c) = 1 + 2c + bc$$

Totally Mixed Nash Equilibria

Nash equilibrium: No player could unilaterally improve own payoff.

Need procedure to find *totally mixed* Nash equilibria: those in *interior* of cube.

Using this, can find all Nash equilibria by *stratification*.



Payoffs to each player of own pure strategies are *equal*.

$$(1) \ 9bc - 3b - 6c + 2 = 0, \text{ i.e., } (3b - 2)(3c - 1) = 0$$

$$(2) \ 9ac - 6a - 3c + 2 = 0, \text{ i.e., } (3a - 1)(3c - 2) = 0$$

$$(3) \ 9ab - 3a - 6b + 2 = 0, \text{ i.e., } (3a - 2)(3b - 1) = 0$$

Two totally mixed Nash equilibria: $(1/3, 1/3, 1/3)$ and $(2/3, 2/3, 2/3)$.

The Status Quo: Gambit

Start with cube containing product of simplices.

Look for single root in cube numerically.

Check that root is in product of simplices.

If intersection of linear hypersurfaces is 0-dimensional, it's a single point. So check linear Taylor approximation to system: good enough within cube? Then this is only root in this cube, so done.

Otherwise, subdivide cube into smaller cubes, start again.

Problem: *neither fast nor robust.*

Groebner Bases

Eliminate terms through computations like these:

$$9bc - 3b - 6c + 2 = 0 \quad (1)$$

$$9ac - 6a - 3c + 2 = 0 \quad (2)$$

$$9ab - 3a - 6b + 2 = 0 \quad (3)$$

$$9abc - 3ab - 6ac + 2a = 0 \quad a * (1)$$

$$-9abc + 6ab + 3bc - 2b = 0 \quad -b * (2)$$

$$3ab - 6ac + 3bc + 2a - 2b = 0 \quad (4)$$

$$9ab - 18ac + 9bc + 6a - 6b = 0 \quad 3 * (4)$$

$$-9ab + 3a + 6b - 2 = 0 \quad -(3)$$

$$-18ac + 9bc + 9a - 2 = 0 \quad (5)$$

Grobner Bases (continued)

$$\begin{array}{r} -18ac + 9bc + 9a \qquad -2 = 0 \\ 18ac \qquad -12a \qquad -6c + 4 = 0 \end{array} \qquad \begin{array}{l} (5) \\ 2 * (2) \end{array}$$

$$\begin{array}{r} 9bc - 3a \qquad -6c + 2 = 0 \end{array} \qquad (6)$$

$$\begin{array}{r} 9bc - 3a \qquad -6c + 2 = 0 \\ -9bc \qquad +3b + 6c - 2 = 0 \end{array} \qquad \begin{array}{l} (6) \\ -(1) \end{array}$$

$$\begin{array}{r} -3a + 3b \qquad = 0 \end{array} \qquad (7)$$

Elimination Theory

End up with *triangular sets*:

equation in one variable

$$9c^2 - 9c + 2 = 0$$

equation in two variables

$$3b - 3c = 0$$

equation in three (here two) variables

$$3a - 3c = 0$$

Find roots of univariate equation, *substitute* into next to get another univariate equation, *repeat*.

Homotopy Continuation

Choose a *start system* similar to the desired system, but such that the roots are *easy* to find.

Then *deform* it gradually towards the desired system, computing the *new roots* at each step.

Starting from each old root, *predictor* step aims at initial guess for new root, *corrector* step actually finds it iteratively.

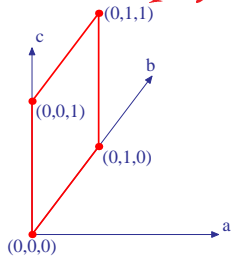
Newton Polytope

Polytope: convex hull of a finite set of points in affine space.

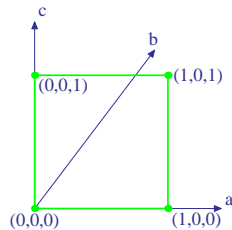
Each monomial $a^\alpha b^\beta c^\gamma \dots$ in n variables is associated with a *lattice point* $(\alpha, \beta, \gamma, \dots) \in \mathbb{N}^n$.

Support of polynomial: monomials occurring with nonzero coefficient.

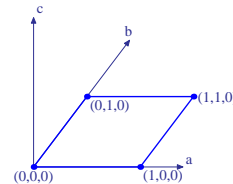
Newton polytope of polynomial: convex hull of lattice points in its support.



$$9bc - 3b - 6c + 2$$



$$9ac - 6a - 3c + 2$$

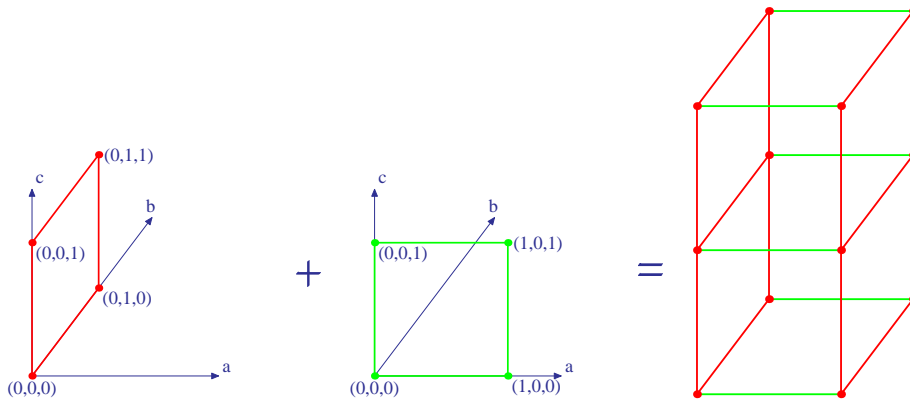


$$9ab - 3a - 6b + 2$$

Minkowski Sum and Mixed Subdivision

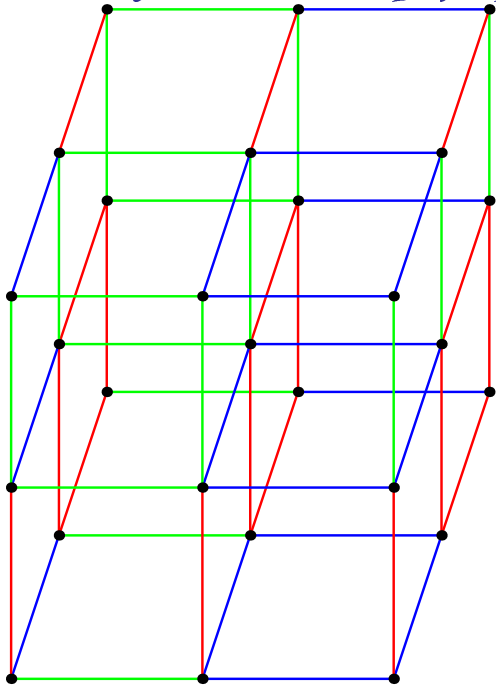
Minkowski sum of polytopes P_1, \dots, P_n is convex hull of $v_1 + \dots + v_n$ where v_i is a vertex of P_i .

Translate faces of P_i along edges of P_j to get decomposition of Minkowski sum into *mixed subdivision* (not unique).



Polyhedral Homotopy Continuation

The number of roots of a generic sparse system of polynomials is given by the *mixed volume* of their Newton polytopes.



Choose a start system
with the *same support*
as the desired system,
so that the roots are easy to find.

Performance

Used Singular for Grobner bases. Used PHC for polyhedral homotopy continuation (a new alternate implementation, PHoM, is parallelizable).

# players	# pure strategies each	# roots	Gambit	Singular	PHC
3	2	2	60-160ms	10ms	20ms
4	2	9	*	70ms	260ms
3	3	10	*	1150ms	350ms
5	2	44	*	**	7s200ms
3	4	56	*	**	13s280ms
6	2	265	*	**	7m10s790ms
4	3	297	*	**	4m3s220ms
3	5	346	*	**	3m19s540ms
3	6	2252	*	**	48m41s870ms
4	4	13833	*	**	7h2m20s780ms