# Research statement for Kiril Datchev

My research is in geometric scattering theory and nonlinear evolution equations. In my current and previous work I have applied microlocal methods to the study of dispersive estimates [Dat09a] and to scattering theory [Dat09b, Dat09c]. In addition, I have studied quantum and semiclassical effects in the evolution of solitary waves [DaHo09, DaVe]. I plan to continue to study the relationship between classical dynamics (such as the trapping of geodesics) and quantum phenomena (such as the high energy behavior of the resolvent of the Laplacian). In particular I will focus on the consequences trapping has for the distribution of resonances near the essential spectrum (§1), and for dispersive estimates (§2). I also plan to begin research in a new but closely related area, namely that of geometric inverse problems (§3).

## 1. Scattering theory and resonances

Resonances are the replacement of discrete spectral data on noncompact domains. Just as an eigenvalue has an associated eigenstate, so a resonance has an associated resonant state. The frequency of oscillation of the resonant state is given by the real part of the resonance, and the rate of decay is given by the imaginary part. My project is to generalize two theorems (Theorems 1.1 and 1.3 below) in the following setting:

**Motivating example.** Let $X = \Gamma\backslash\mathbb{H}$ (here $\mathbb{H}$ is the hyperbolic plane and $\Gamma$ a discrete subgroup of isometries) be a convex-cocompact finitely generated Riemann surface, that is a constant negative curvature surface with infinite funnel ends. Two important objects in that setting are the *Eisenstein series* $E_\Gamma(s)$, and the *Selberg zeta function*, $Z_\Gamma(s)$. Each is initially defined on $\{\operatorname{Re} s > 1/2\}$, but has a meromorphic continuation to $\mathbb{C}$. The poles of $E_\Gamma(s)$ and the zeros of $Z_\Gamma(s)$ are called (in a more general setting) resonances. An important dynamical object associated to $\Gamma$ is the limit set $\Lambda(\Gamma) \subset \partial\mathbb{H}$. Let $\delta(\Gamma)$ be the Hausdorff dimension of $\Lambda(\Gamma)$.

For more general manifolds $(X, g)$, define *resonances* to be poles of the meromorphic continuation of the cutoff resolvent $\chi(-\Delta_g - s(1-s))^{-1}\chi$ from $\{\operatorname{Re} s > 1/2\}$ across the essential spectrum $\{\operatorname{Re} s = 1/2\}$. In place of the limit set I use the *trapped set* $K \subset T^*X$ of maximally extended geodesics which are precompact. This is related to $\Lambda$ in the case of the example by Patterson-Sullivan theory [Pat76, Sul79]; in particular $\dim K = 2\delta + 2$. Assume negative $X$ has curvature, so that $K$ is hyperbolic. I also use the *topological pressure* $\mathcal{P}(\alpha)$, which I do not define here, but in the example above $\mathcal{P}(\alpha) = \delta - \alpha$. My methods are valid in some higher-dimensional settings as well, but for simplicity I state them in dimension two here.

**Theorem 1.1** (Resonance free strip, [Pat76, Sul79, Pat88])**.** *Let $X = \Gamma\backslash\mathbb{H}^2$ be a convex-cocompact Riemann surface, and let $\delta$ be the dimension of its limit set. If $\delta < 1/2$, then $X$ has a simple resonance at $s = \delta$, and no other resonances in $\{\operatorname{Re} s \geq \delta\}$.*

In other words having $\delta < 1/2$ implies a gap between the resonances and the essential spectrum $\{\operatorname{Re} s = 1/2\}$. Starting with the work of [Dat09b], I proved

**Theorem 1.2** ([Dat09c])**.** *Let $\tilde{X}$ be a compactly supported metric perturbation of $X$ with negative curvature. Suppose the topological pressure $\mathcal{P}(\alpha)$ of the trapped set $K$ obeys $\mathcal{P}(1/2) < 0$. Then for any $M_2 < -\mathcal{P}(1/2)$ there exists $M_3 > 0$, such that $\Delta_g$ has no resonances in the region $\{s \in \mathbb{C} \colon |\operatorname{Im} s| > M_3, \operatorname{Re} s > 1/2 - M_2\}$.*

In this setting the size of the gap is known only at high energies, and there is a loss of size $\varepsilon$ in the width of the gap. In [NoZw], Nonnenmacher-Zworski, extending work of Ikawa [Ika88] and using methods of Anantharaman [Ana08], prove the analog of Theorem 1.2 in an asymptotically Euclidean setting. Such gaps have been studied in the physics literature at least since the seminal work of Gaspard-Rice [GaRi89].

To study resonances in [Dat09b, Dat09c] I use the method of *complex scaling* of Aguilar-Combes [AgCo71] and Simon [Sim72], following the geometric approach of Sjöstrand-Zworski [SjZw91]. In this method one holomorphically deforms $\Delta_g$ into a nonselfadjoint operator $\Delta_\theta$ with eigenvalues where $\Delta_g$ has resonances. A computational implementation of a similar approach is known as Perfectly Matched Layers (PML) in numerical analysis, which has many applications, including the study of Microelectromechanical Systems [Ber94, BiGo05] (see [Dat06] for an expository treatment of the relationship between complex scaling and PML). So far this method has resisted application to the case of asymptotically hyperbolic spaces. In my results I require a warped product structure of the metric near infinity.

In the case where $\delta \geq 1/2$, or more generally when $\mathcal{P}(1/2) \geq 0$, there is a bound on the density of resonances in terms of $m = \dim K$.

**Theorem 1.3** (Fractal Weyl upper bound, [Zwo99, GLZ04])**.** *Suppose $X = \Gamma\backslash\mathbb{H}$ is a convex cocompact Riemann surface. Let $R(X)$ denote the set of resonances of $X = \Gamma\backslash\mathbb{H}$, included with multiplicity, and let $m$ be the dimension of the trapped set $K \subset T^*X$. Then, for any $C_0 > 0$ there exists $C_1 > 0$ such that for all $r > 1$,*

$$\#\{s \in R(X)\colon |\operatorname{Im} s| \leq r,\, \operatorname{Re} s \geq -C_0\} \leq C_1 r^{m/2}. \tag{1.1}$$

The terminology comes from the fact that this is a partial generalization to the case of resonances of the Weyl asymptotic for eigenvalues of a compact manifold. Starting with the work of [Dat09b] I proved

**Theorem 1.4** ([Dat09c])**.** *Let $\tilde{X}$ be a compactly supported metric perturbation of $X$ which has negative curvature. Let $m$ be the Minkowski dimension of the trapped set $K \subset T^*X$, and let $R(\tilde{X})$ denote the set of resonances, included with multiplicity. Then for any $M_2 > 0$, $\widetilde{m} > m$ there exists $C > 0$ such that*

$$\#\{s \in R(\tilde{X})\colon |\operatorname{Im} s| \leq r, \operatorname{Re} s > (n-1)/2 - M_2\} \leq C r^{\widetilde{m}/2},$$

*and when $K$ is of pure dimension, $\widetilde{m}$ may be replaced by $m$.*

The first upper bounds of this type appear in Sjöstrand [Sjö90]; see also Wunsch-Zworski [WuZw00] and Sjöstrand-Zworski [SjZw07] for further developments. Lower bounds have proved more elusive, but the geometric setting holds more promise by offering richer structure. This is indicated by recent work of Jakobson-Naud [JaNa], who obtain polynomial lower bounds (with exponent related to $\dim K$) on the density of resonances for certain arithmetic convex cocompact quotients. One then hopes for an eventual *Fractal Weyl asymptotic*, although for now this is more speculative. Fractal Weyl laws, which originated in the mathematical PDE community, have attracted physics attention in recent years [Phys].

**Question 1.5.** For what classes of infinite ends are the results of Theorems 1.2 and 1.4 valid?

One case which appears especially accessible is that of a *scattering manifold* which is real analytic outside of a compact set, extending the result of [WuZw00]. Scattering manifolds form a class of asymptotically Euclidean spaces, introduced by Melrose in [Mel94].

In [WuZw00] the method of complex scaling is adapted to scattering manifolds which are globally real analytic, but the smooth gluing method of [Dat09b, Dat09c] should work in this setting also and allow me to assume that $X$ is only analytic outside of a compact set.

An interesting open problem is to generalize Theorems 1.3 and 1.4 to the case of cusps, in other words to the case where $\Gamma\backslash\mathbb{H}$ is not assumed to be convex cocompact. If $\Gamma\backslash\mathbb{H}$ is a surface with cusps we have always $\delta > 1/2$, and as a result $\mathcal{P}(1/2) > 0$, and so a generalization of Theorem 1.1 in the style of Theorem 1.2 seems unlikely. However, the statement of Theorem 1.4 stil makes sense, giving the following

**Question 1.6.** Let $X = \Gamma\backslash\mathbb{H}$ be a finitely generated hyperbolic surface, and let $m$ be the dimension of its trapped set. Can we show that for any $C_0 > 0$ there is $C_1 > 0$ such that

$$\#\{s \in R(X)\colon\ \operatorname{Im} s \le r,\ \operatorname{Re} s \ge -C_0\} \le C_1 r^{(1+m)/2}?$$

If $\Gamma\backslash\mathbb{H}$ has funnels but no cusps, this is Theorem 1.3, and if it has cusps but no funnels, this is a consequence of work of Selberg [Sel]. It is natural to conjecture the same statement when $\Gamma\backslash\mathbb{H}$ has both cusps and funnels. Towards that I have proved the following

**Theorem 1.7** ([Dat09b])**.** *Let* $\Gamma$ *be generated by* $z \mapsto z + 1$*, and let* $X$ *be a nonpositively curved, compactly supported metric perturbation of* $\Gamma\backslash\mathbb{H}$*. Then for any* $M_2 > 0$ *there exists* $M_3 > 0$ *such that* $\Delta_g$ *has no resonances in the region*

$$\{s \in \mathbb{C} : |\operatorname{Im} s| > M_3, \operatorname{Re} s > -M_2 \log|\operatorname{Im} s|\}. \tag{1.2}$$

Here $\Gamma\backslash\mathbb{H}$ is the parabolic cylinder, a surface of revolution with one cusp end and one funnel end. In this setting the nonpositive curvature assumption implies that the manifold is nontrapping.

Logarithmic resonance free regions such as the one in Theorem 1.7 go back to work of Lax and Phillips [LaPh] and Vainberg [Vai], with developments by Sjöstrand [Sjö90], Martinez [Mar02], and Sjöstrand-Zworski [SjZw07]. To my knowledge Theorem 1.7 is the first such result for manifolds with cusps.

Recently, work of Dolgopyat [Dol98] in dynamical systems has made it possible to deduce larger resonance free strips. In [Nau05] Naud shows that if $X$ is a hyperbolic quotient with $\delta < 1/2$, then there exists $\varepsilon > 0$ such that the simple pole at $s = \delta$ in Theorem 1.1 is the only resonance in the half plane $\operatorname{Re} s > \delta - \varepsilon$. In [PeSt], Petkov and Stoyanov study the several obstacle problem in $\mathbb{R}^n$, and establish the existence of a strip of width $-\mathcal{P}(1/2) + \varepsilon$ with only finitely many resonances. This suggests the following, possibly subtle,

**Question 1.8.** Is there also a wider resonance free strip in the setting of Theorem 1.2 and of [NoZw]?

## 2. DISPERSIVE PARTIAL DIFFERENTIAL EQUATIONS

2.1. **Dispersive estimates.** The distribution of resonances is related to dispersive PDE by resolvent estimates. Using propagation of singularities and microlocal gluing methods together with results of Vasy-Zworski [VaZw00], Cardoso-Vodev [CaVo02] and Nonnenmacher-Zworski [NoZw], in [Dat09a] I prove the following result for scattering manifolds (these are a general class of asymptotically Euclidean manifolds introduced by Melrose in [Mel94]):

**Theorem 2.1** ([Dat09a])**.** *Let* $(X, g)$ *be a scattering manifold which has geodesic flow satisfying* $\mathcal{P}(1/2) < 0$ *on its trapped set, then for any* $\chi \in C_0^\infty(X)$*,* $\lambda_0 > 0$*, there exists* $C$ *such* *that, for* $\lambda \geq \lambda_0$,

$$\left\|\chi(\Delta_g - \lambda - i0)^{-1}\chi\right\|_{L^2(X)\to L^2(X)} \leq C\frac{\log(1+\lambda)}{\sqrt{\lambda}}. \tag{2.1}$$

In the nontrapping case $\log(1+\lambda)$ can be replaced by 1 (see [CPV04] for a general statement and more references). Such a resolvent estimate leads to information about the Schrödinger propagator, see for example Burq-Gérard-Tzvetkov [BGT04] for a $TT^*$ argument and applications to nonlinear Schrödinger equations.

The following local smoothing inequality is a consequence of (2.1):

$$\int_0^T \left\|\chi e^{it\Delta_g} u\right\|^2_{H^{\frac{1}{2}-\eta}(X)} dt \leq C_{\eta,T}\|u\|^2_{L^2(X)}, \qquad \eta > 0. \tag{2.2}$$

Estimates of this type go back to work of Sjölin [Sjö87], Vega [Veg88], and Constantin-Saut, [CoSa88]. Doi [Doi96] shows that in a wide variety of geometric settings the absence of trapped geodesics is necessary for (2.2) to hold with $\eta = 0$. In other words trapping implies a loss in the local smoothing effect. However, in [BGH], Burq-Guillarmou-Hassell show that (2.1) implies the following lossless Strichartz estimate

$$\|e^{it\Delta_g}u_0\|_{L^p((0,1),L^q(X))} \leq C\|u_0\|_{L^2(X)}, \qquad p > 2, \quad 2/p + n/q = n/2. \tag{2.3}$$

**Question 2.2.** What assumptions on the infinite ends of $X$ are needed to prove (2.1) and (2.3), provided the trapped set $K$ is hyperbolic and the topological pressure obeys $\mathcal{P}(1/2) < 0$?

I plan to build on methods of [NoZw], [BGH] and [Dat09c] to deduce (2.1) and (2.3) from Theorem 1.2 in the case where the manifold is a compactly supported, negatively curved perturbation of a convex-cocompact hyperbolic quotient. This will extend to the case of nonconstant curvature a result of [BGH], where (2.3) is proved for an exact convex-cocompact hyperbolic quotient.

In [CPV04] Cardoso-Popov-Vodev study a very general class of infinite ends, which includes many kinds of asymptotically Euclidean and asymptotically hyperbolic manifolds. To treat that class of ends I will prove a microlocal version of their resolvent estimate near infinity. Where they use only cutoffs in space together with integration by parts, I will incorporate a microlocal cutoff in phase space, distinguishing the incoming and outgoing regions of phase space. This will allow me to prove a propagation of singularities result, and to apply the cutoff function methods of [Dat09a].

## 2.2. **Nonlinear dispersive equations.**

I am interested not only in the derivation of dispersive estimates, but also in their applications to the study of nonlinear equations, and in particular to *solitons*. In [DaHo09] and [DaVe] I have studied the following two nonlinear Schrödinger equations with external potential:

$$iu_t = -u_{xx}/2 - |u|^2 u + V(x)u, \qquad u\colon \mathbb{R}\times\mathbb{R}\to\mathbb{C}, \tag{2.4}$$

$$iu_t = -\Delta u/2 - (|x|^{-1} * |u|^2)u + V(x)u, \qquad u\colon \mathbb{R}\times\mathbb{R}^3\to\mathbb{C}. \tag{2.5}$$

The first equation is called the Gross-Pitaevskii equation, and the second the Hartree equation. Both model Bose-Einstein condensates formed by large collections of bosons at very low temperature.

In the Gross-Pitaevskii equation (2.4), the ground state soliton of the free equation is given by $\operatorname{sech}(x)$. With Justin Holmer we prove the following soliton splitting result:

**Theorem 2.3** ([DaHo09]). *Let $q \in \mathbb{R}$, $V(x) = q\delta_0(x)$, $0 < \varepsilon \ll 1$. Let $u(t,x)$ be the solution of (2.4) with $V(x) = q\delta_0(x)$, with initial condition $u(0,x) = e^{ixv}\operatorname{sech}(x - x_0)$ such that $x_0 \leq -v^{-\varepsilon}$. Then for $t \in (v^{-1}|x_0| + 1, \varepsilon \log v)$, we have, as $v \to \infty$ with $|q| \sim v$,*

$$u(t,x) = u_T(t,x) + u_R(t,x) + \mathcal{O}_{L^\infty_x}((t - |x_0|/v)^{1/2}) + \mathcal{O}_{L^2_x}(v^{-1/2+\varepsilon}),$$

$$u_T(t,x) = e^{i\phi_T} e^{ixv + i(A_T^2 - v^2)t/2} A_T \operatorname{sech}(A_T(x - x_0 - tv)),$$

*where the amplitude $A_T$ and the phase $\phi_T$ are explicitly given. We also have a similar expression for $u_R$.*

In other words, we show that an incident soliton at high velocity splits into two parts, a transmitted and a reflected part, which we describe explicitly using the Zakharov-Shabat [ZaSh72] inverse scattering theory. The evolution is governed by the scattering matrix of the operator $-(1/2)(d^2/dx^2) + q\delta_0(x)$. In particular, even when $q < 0$, that is to say when we have an attractive delta potential, the soliton is split in two and repelled in the high velocity limit. In particular, the trapping caused by the nonlinear eigenstate $u(t,x) = e^{i\lambda^2 t/2}\lambda \operatorname{sech}(\lambda|x| + \tanh^{-1}(|q|/\lambda))$ is negligible. The main novelty in our paper [DaHo09] compared to previous work by Holmer-Marzuola-Zworski [HMZ07] is the treatment of the case $q < 0$. The external potential given by a delta function has also been studied in the physics literature by Cao-Malomed [CaMa95].

A possible direction for future work is indicated by recent work of Abou-Salem and Sulem [AbSu], who study soliton tunneling for a *double delta potential* $V(x) = q(\delta_0(x+l) + \delta_0(x-l))$:

**Question 2.4.** What kinds of soliton splitting occur in (2.4) when $V(x)$ is a double delta potential?

Let $\eta$ denote the ground state soliton of (2.5). With Ivan Ventura we exploit a breakthrough in the spectral theory of the linearized Hartree operator from Lenzmann [Len09] to extend work of Holmer-Zworski [HZ08]. We show that in the presence of a slowly varying external potential $V(x) = W(hx)$ and up to an error of size $h^2$, a soliton retains its shape and evolves according to an effective classical Hamiltonian:

**Theorem 2.5** ([DaVe]). *Let $V(x) = W(hx)$, where $W \in C^3(\mathbb{R}^3; \mathbb{R})$. Fix $c_1 > 0$, $(v_0, a_0) \in \mathbb{R}^3 \times \mathbb{R}^3$, and let $u(t,x)$ be the solution of (2.5) with initial condition $u(0,x) = e^{ixv}\eta(x - a_0)$. There exist constants $h_0$ and $c_2$ such that for all $0 < \varepsilon \ll 1$ and $0 < h \leq h_0$ and $t \in [0, c_1/h + \delta|\log h|/(c_2 h)]$, we have*

$$\left\| u(t,x) - e^{v(t)\cdot(x - a(t))} e^{i\gamma(t)} \eta\left[(x - a(t))\right] \right\|_{H^1_x(\mathbb{R}^3)} \leq c_2 h^{2-\varepsilon}.$$

*Here $(a, v, \gamma)$ solve, with initial data $(a_0, v_0, 0)$, the following system of equations of motion*

$$\dot{a} = v, \quad \dot{v} = -\frac{1}{2}\int \nabla V\,(x + a)\,\eta^2(x)dx,$$

$$\dot{\gamma} = \frac{1}{2}|v|^2 + \lambda - \frac{1}{2}\int V\,(x + a)\,\eta^2(x)dx + \frac{1}{2}\int x \cdot \nabla V\,(x + a)\,\eta^2(x)dx.$$

Our result improves those of earlier authors who study the same problem in a more general setting, including Fröhlich-Tsai-Yau [FTY02], Fröhlich-Gustafson-Jonsson-Sigal [FGJS04] and Abou-Salem [Abou08], in that we provide a more precise error bound and we provide exact equations of motion. The phenomenon of effective classical dynamics in NLS has been of interest to the physics community as well (see e.g. [EbMa83]).

In [JFGS06], Jonsson-Fröhlich-Gustafson-Sigal study the case of a *confining* potential, and obtain effective equations of motion for a longer time, namely up to time $h^{-3/2}$ rather than time $h^{-1}\log(1/h)$. However, they do not obtain the improvements mentioned in the previous paragraph. This leads to the following

**Question 2.6.** What improvements are possible to Theorem 2.5 in the case of a confining potential?

Combining the methods of [DaVe] and [JFGS06] should give at least a partial answer.

## 3. CALDERÓN'S INVERSE PROBLEM

A new direction in which my expertise will be useful is that of Calderón's inverse problem. To describe it, let $\Omega \subset \mathbb{R}^n$ be bounded and open with smooth boundary, and let $\gamma \in C^\infty(\overline{\Omega}; (0,\infty))$. For each $f \in C^\infty(\partial\Omega)$, the equation $\nabla \cdot \gamma \nabla u = 0$ has a unique solution in $\Omega$ satisfying $u|_{\partial\Omega} = f$. Let $g = \partial_\nu u|_{\partial\Omega}$, where $\partial_\nu$ is the outward normal derivative of $u$ on $\partial\Omega$. Define the *Dirichlet-to-Neumann* map $\Lambda_\gamma \colon C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ by $\Lambda_\gamma(f) = g$. Calderón's inverse problem [Cal80] asks if the map $\gamma \mapsto \Lambda_\gamma$ is injective. As stated above the question is answered in the affirmative by Sylvester and Uhlmann [SyUh87] in the case $n \geq 3$, and by Nachman [Nac96] in the case $n = 2$. Interesting unsolved problems remain if one is allowed only *partial data*:

**Question 3.1** (Calderón's problem with partial data)**.** Let $U, V \subset \partial\Omega$ be open, and suppose $\gamma|_{\partial\Omega}$ is known. Suppose further $\chi_2 \Lambda_\gamma \chi_1 \colon C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$ is known for every $\chi_1 \in C_0^\infty(U)$ and $\chi_2 \in C_0^\infty(V)$. Does this information determine $\gamma$?

One physical interpretation of this question is as follows. Let $\gamma$ be the electrical conductivity of the object $\Omega$. Can we determine $\gamma$ by applying voltage in the region $U$ and measuring the resulting current at $V$?

The results of [SyUh87] address the case $n \geq 3$ and $U = V = \partial\Omega$. Bukhgeim-Uhlmann [BuUh02] and Kenig-Sjöstrand-Uhlmann [KSU07] show that the answer is yes in the case $n \geq 3$, provided $U$ and $V$ are not both very small. For $n = 2$, Imanuvilov-Uhlmann-Yamamoto [IUY] are able to show that the answer is yes so long as $U = V \neq \varnothing$. The set $U$ in this case is arbitrarily small. It is natural to ask whether $U$ and $V$ can be made small in the case $n \geq 3$.

This is a difficult open problem, and I propose to begin by solving the following simpler problem. To state it, let $q = \gamma^{-1/2}\Delta(\gamma^{1/2})$, and let $\Lambda_q$ be the Dirichlet-to-Neumann map associated to the operator $-\Delta + q$. Then the injectivity of $\gamma \mapsto \Lambda_\gamma$ is equivalent to that of $q \mapsto \Lambda_q$. Finally if $U = V$, then $q_1$ and $q_2$ have $\Lambda_{q_1} = \Lambda_{q_2}$ if and only if $\int_\Omega (q_1 - q_2) u_1 u_2 dx = 0$ for all $u_1$ and $u_2$ obeying $u_1|_{\partial\Omega\setminus U} = u_2|_{\partial\Omega\setminus U} = 0$, and $(-\Delta + q_j)u_j = 0$. Consider the associated *linearized problem*:

**Question 3.2** (Linearized Calderón's problem with partial data)**.** Fix $q_0 \in C^\infty(\overline{\Omega})$ and $U \subset \partial\Omega$ open. Suppose $\int_\Omega q u_1 u_2 dx = 0$ for all $u_1$ and $u_2$ solving $(-\Delta + q_0)u_j = 0$, and obeying $u_1|_{\partial\Omega\setminus U} = u_2|_{\partial\Omega\setminus U} = 0$. Does it follow that $q \equiv 0$? In other words, do the possible products $u_1 u_2$ constitute a dense subest of $L^1(\Omega)$?

Dos Santos Ferreira-Kenig-Sjöstrand-Uhlmann [DKSU] prove that they do in the case $q_0 \equiv 0$, no matter how small $U$ is, and for all dimensions $n \geq 2$. I plan to treat the case $q_0 \not\equiv 0$ in collaboration with Hamid Hezari and Michael VanValkenburgh. The first part of the proof in [DKSU] is a local-to-global argument based on Runge's approximation theorem, and this generalizes without difficulty to the case $q_0 \not\equiv 0$. This reduces the problem to finding a neighborhood of $\Omega'$ of $U$ in $\Omega$ such that the products $u_1 u_2$ are dense in $L^1(\Omega')$. To do this, the authors construct a family of complex geometric optics solutions $u_\zeta(x)$ with $\zeta \in \mathbb{C}$ of the form $u_\zeta(x) = e^{-ix\cdot\zeta/h} + w_\zeta(x)$ (here $h$ is a semiclassical parameter), and use an argument based on the *Kashiwara watermelon theorem* for analytic microlocal singularities to show that the products $u_{\zeta_1} u_{\zeta_2}$ are dense in $L^1(\Omega')$. For the case $q_0$ nonzero we will have an additional correction term $\tilde{w}_\zeta$ in the spirit of [BuUh02, KSU07], and a closer analysis will be needed.

Another interesting case to consider, for which there are no results in either the linear or nonlinear setting, is $U \cap V = \varnothing$. The linearized problem is likely easier, and this is a natural next step after Question 3.2.

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