Workshop on Spectral Theory and Harmonic Analysis, ANU, Canberra

Solitary wave dynamics in a slowly varying potential.

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UC Berkeley

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In this talk we will study the focusing nonlinear Schrödinger equation (NLS)

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We will look at two examples:

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In both examples above, the nonlinear term acts as an attractive effective potential, which causes the focusing effect.

Physical interpretation

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In the case $F(|u|^2) = |x|^{-1} * |u|^2$, $n = 3$, the equation (1) is called the Hartree equation. This equation models boson gases at low temperature when the gas particles interact by the Coulomb electrostatic force: this is the source of the $|x|^{-1}$ in F.

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For the remainder of the talk, for definiteness, we will focus on the Hartree case, although many analogous results are true for the Gross-Pitaevskii and for other nonlinearities.

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From this soliton state we have a family of solutions

$$e^{ix\cdot v}e^{i|v|^2t/2}e^{i\gamma}e^{i\lambda t}\mu^2\eta(\mu(x-a-vt))$$

to

$$i\partial_t u = -\frac{1}{2}\Delta u - (|x|^{-1} * |u|^2)u.$$

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Notice the center of mass obeys a classical equation of motion.

In this talk we will see how adding an external potential $V\colon \mathbb{R}^3 \to \mathbb{R}$ can give a richer example of the same phenomenon:

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Compare this to

$$u(x,t) = e^{ix\cdot v} e^{i|v|^2 t/2} e^{i\gamma} e^{i\lambda t} \mu^2 \eta(\mu(x - a - vt))$$

in the case $V \equiv 0$.

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In a recent preprint available on `arXiv`, Ivan Ventura and I prove the following

Theorem. Let $V(x) = W(hx)$ with $W \in C^3$. Fix $0 < c_1$ and $(v_0, a_0) \in \mathbb{R}^3 \times \mathbb{R}^3$. Suppose u solves

$$i\partial_t u = -\frac{1}{2}\Delta u - (|x|^{-1} * |u|^2)u + V(x)u, \quad u(0) = e^{iv_0(x-a_0)}\eta(x - a_0).$$

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Then for

$$0 \leq t \leq \frac{c_1}{h} + \frac{\delta |\log h|}{c_2 h}, \qquad , 0 \leq \delta < 1/2 \quad 0 < h \leq h_0,$$

we have

$$\|u(x,t) - e^{iv(t)(x-a(t))}e^{i\gamma(t)}\eta(x - a(t))\|_{H^1} \leq c_2 h^{2-\delta}.$$

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(We will see presently that the natural unit time scale in this problem is h^{-1}.)

In the estimate

$$\|u(x,t) - e^{iv(t)(x-a(t))}e^{i\gamma(t)}\eta(x-a(t))\|_{H^1} \le c_2 h^{2-\delta},$$

the functions $(a(t), v(t), \gamma(t))$ solve

$$\dot{a} = v, \qquad \dot{v} = -\frac{1}{2}\int \nabla V\,(x+a)\,\eta^2(x)dx,$$

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Moreover,

$$-\frac{1}{2}\int \nabla V\,(x+a)\,\eta^2(x)dx = -\nabla V(a) + \mathcal{O}(h^3).$$

In summary, up to time $t \sim h^{-1} + \delta |\log h| h^{-1}$, we have

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Remark. The same result is valid if we take

$$\|u(0) - e^{iv_0(x-a)} \eta(x - a_0)\|_{H^1} \leq c_1 h^2.$$

We also have a slightly weaker result in the case

$$\|u(0) - e^{iv_0(x-a)} \eta(x - a_0)\|_{H^1} \leq c_1 h^{1/2+\delta_0}.$$

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$$\dot{a} = v + \mathcal{O}(h^2), \quad \dot{v} = -\nabla V(a) + \mathcal{O}(h^2),$$

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- More precise error estimates. Where previous authors had

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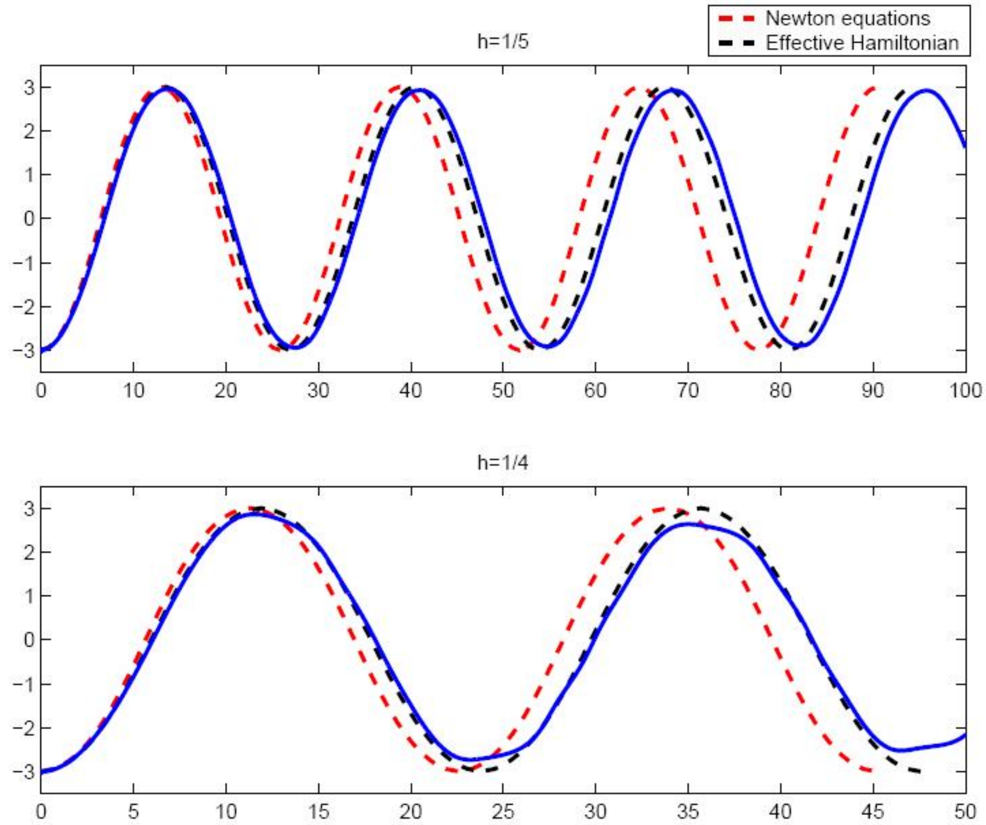

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We have used here

$$\omega(v, \Xi_{H_V}(u)) = d_u H(v), \qquad \partial_t u = \Xi_{H_V}(u).$$

Manifold of solitons.

We define the manifold of solitons $M \subset H^1$ to be the set

$$M = \{e^{iv(x-a)}e^{i\gamma}\mu^2\eta(\mu(x-a)); (a,v,\gamma,\mu) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^+\}.$$

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In (a, v, γ, μ) coordinates we have

$$\omega\big|_M = \mu dv \wedge da + v d\mu \wedge da + d\gamma \wedge d\mu,$$

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It is from this that we obtain the equations of motion

$$\dot{a} = v, \quad \dot{v} = -\frac{1}{2}\int \nabla V\,(x+a)\,\eta^2(x)dx, \qquad \dot{\gamma} = ...$$

In summary, in order to determine the equations of motion for (a, v, γ) in

$$\|u(x,t) - e^{iv(t)(x-a(t))}e^{i\gamma(t)}\eta(x-a(t))\|_{H^1} \leq c_2 h^{2-\delta},$$

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- We take the corresponding Hamiltonian function H_V and restrict it to the manifold of solitons $M = \{e^{iv(x-a)}e^{i\gamma}\mu^2\eta(\mu(x-a))\}$.

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$$\partial_t u = -i\left(-\frac{1}{2}\Delta u - (|x|^{-1} * |u|^2)u + V(x)u\right) = \Xi_{H_V}(u)$$

- We take the corresponding Hamiltonian function H_V and restrict it to the manifold of solitons $M = \{e^{iv(x-a)}e^{i\gamma}\mu^2\eta(\mu(x-a))\}$.
- We posit that (a, v, γ) should evolve according to the Hamilton flow of the restricted Hamiltonian.

In summary, in order to determine the equations of motion for (a, v, γ) in

$$\|u(x,t) - e^{iv(t)(x-a(t))}e^{i\gamma(t)}\eta(x-a(t))\|_{H^1} \leq c_2 h^{2-\delta},$$

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  (μ ends up varying so little that we can neglect it).

Ideas from the proof.

Using the implicit function theorem, we write (initially for small time)

$$u(t) = \tilde{u}(t) + w(t),$$

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It then remains to prove that w is small.

Estimating w.

Recall that w is the difference between u, the true solution, and M, the manifold of solitons.

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To do this we employ a variant of the Lyapounov functional, used by Michael Weinstein (1986):

$$\|w\|^2_{H^1_x} \leq c\mathrm{Re}\int \mathcal{L}w\bar{w}dx, \tag{2}$$

where $\mathcal{L}$ is the linearization of the Hartree differential operator.

$$\mathcal{L}w \stackrel{\text{def}}{=} -\frac{1}{2}\Delta w - \left(\frac{1}{|x|} * \eta(w+\bar{w})\right)\eta - \left(\frac{1}{|x|} * \eta^2\right)w + \lambda w.$$

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That the estimate (2) is valid for w symplectically orthogonal to M is a result of Enno Lenzmann (2008).

We build most closely off of work of Holmer-Zworski (2007), who prove a similar theorem for the cubic NLS in 1 dimension:

$$i\partial_t u = -\frac{1}{2}\partial_x^2 u - |u|^2 u + V(x)u.$$

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- We no longer have an explicit formula (namely $\eta(x) = \operatorname{sech} x$) for the soliton. As a result of this we must use more abstract arguments to study our linearized operator $\mathcal{L}$, including crucially the work of Lenzmann.
- We study more general initial data. Where H-Z required

$$\|u(0) - e^{iv_0(x-a)}\eta(x-a_0)\|_{H^1} \leq c_1 h^{3/2+\delta_0},$$

we only require

$$\|u(0) - e^{iv_0(x-a)}\eta(x-a_0)\|_{H^1} \leq c_1 h^{1/2+\delta_0}.$$

Extensions to other nonlinearities.

This phenomenon can also be observed for other nonlinearities: Fröhlich-Tsai-Yau (2002), Fröhlich-Gustafson-Jonsson-Sigal (2004), and Abou-Salem (2008) study general equations of the form

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This assumption is the most delicate, and poses the most substantial barrier to the further extension of these results.

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