

AMS Meeting, University of New Mexico

Fast soliton splitting by an attractive delta potential.

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UC Berkeley

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Suppose now that the initial condition is left unchanged, while the equation is **perturbed** by a delta potential:

$$i\partial_t u = -\frac{1}{2}\partial_x^2 u - |u|^2 u + q\delta_0(x)u,$$

where $q \in \mathbb{R}$ is a *coupling constant*.

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Over the last two years, numerical and analytic work by Holmer-Marzuola-Zworski and Holmer-Zworski has described the interaction of the soliton with the potential under various assumptions on the **speed of the soliton v** and the **size of the coupling constant q** .

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This talk will focus on the high velocity case $v \gg 1$, and we will consider both attractive and repulsive potentials ($q < 0$ and $q > 0$).

(Of course this list of previous work is not exhaustive!)

Recall

$$u_0(x) = e^{ixv} \operatorname{sech}(x - x_0), \quad i\partial_t u = -\frac{1}{2}\partial_x^2 u - |u|^2 u - q\delta_0(x)u.$$

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For $x_0 \ll -1$, $v > 0$, we expect the solution to essentially remain the rightward propagating soliton

$$u(t, x) = e^{-itv^2/2} e^{ixv} \operatorname{sech}(x - x_0 - vt),$$

until time $t \sim |x_0|/v$.

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Theorem. Let $\varepsilon > 0$. Suppose $|x_0|/v + 1 \leq t \leq \varepsilon \log v$. Then, as $v \rightarrow \infty$,

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where

$$u_T(t, x) = e^{i\phi_T} e^{ixv} e^{i(A_T^2 - v^2)t/2} A_T \operatorname{sech}(A_T(x - x_0 - vt)),$$

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Here ϕ_T, A_T, ϕ_R, A_R depend on v and q , but not x or t or ε . We have $\sigma = -1$ for $q > 0$ and $\sigma = -5/6$ for $q < 0$.

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H-M-Z (2007) proved a similar theorem for $q > 0$.

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Then, as $v \rightarrow \infty$,

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The same method is likely to give a similar (although weaker) result in the case $q < 0$, but the calculations are prohibitively

complicated...

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Suppose

$$i\partial_t u + \frac{1}{2}\partial_x^2 u - q\delta_0(x)u = f(t, x), \quad u(0, x) = u_0(x),$$

for $t \in [0, T]$.

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Then, for $q \geq 0$, and for $p, r, \tilde{p}', \tilde{r}'$ suitably related,

$$\|u\|_{L_t^p L_x^r} \leq C\|u_0\|_{L_x^2} + C\|f\|_{L_t^{\tilde{p}'} L_x^{\tilde{r}'}}.$$

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For $q < 0$, the dispersion is imperfect, and we have instead

$$\|u\|_{L_t^p L_x^r} \leq C\|u_0\|_{L_x^2} + CT^{1/p}\|Pu_0\|_{L_x^r} + C\|f\|_{L_t^{\tilde{p}'} L_x^{\tilde{r}'}} + CT^{1/p}\|Pf\|_{L_t^1 L_x^r},$$

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Crucially, the constant C does not depend on q .

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This is proved by multiplying the equation by $\partial_t \bar{u}$, integrating in space and time, and taking the real part.

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We use a single, smaller piece, of size $v^{-1+\varepsilon}$ for the interval in which the soliton interacts with the potential. (Note the error in the above image!)

In the pre-interaction phase, we use the Strichartz estimate to control the error, one time interval at a time.

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$$u(t) = e^{i(t-t_1)H_q}u(t_1) + \int_{t_1}^t e^{i(t-s)H_q}|u(s)|^2u(s)ds.$$

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The first term is linear, and hence easier to analyze. The second term is small because $|t - t_1| \lesssim v^{-1+\varepsilon}$.

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As a result we are in a position avoid an accumulation of errors

in the post-interaction phase when $q < 0$, and are able to improve the transmission result for $q > 0$.

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Now when we use the Strichartz estimate, the second term is **eliminated**:

$$\|u\|_{L_t^p L_x^r} \leq C \|u_0\|_{L_x^2} + CT^{1/p} \|Pu_0\|_{L_x^r} + C \|f\|_{L_t^{\tilde{p}'} L_x^{\tilde{r}'}} + CT^{1/p} \|Pf\|_{L_t^1 L_x^r}.$$

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