

LEFSCHETZ HYPERPLANE THEOREM FOR STACKS

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ABSTRACT. We use Morse theory to prove that the Lefschetz Hyperplane Theorem holds for compact smooth Deligne-Mumford stacks over the site of complex manifolds. For $\mathfrak{Z} \subset \mathfrak{X}$ a hyperplane section, \mathfrak{X} can be obtained from \mathfrak{Z} by a sequence of deformation retracts and attachments of high-dimensional finite disc quotients. We use this to derive more familiar statements about the relative homotopy, homology, and cohomology groups of the pair $(\mathfrak{X}, \mathfrak{Z})$. We also prove some preliminary results suggesting that the Lefschetz Hyperplane Theorem holds for Artin stacks as well. One technical innovation is to reintroduce an inequality of Lojasiewicz which allows us to prove the theorem without any genericity or nondegeneracy hypotheses on \mathfrak{Z} .

1. INTRODUCTION

Consider an n -dimensional complex manifold X with a positive line bundle L and a global section s with zero locus $Z \subset X$. The Lefschetz hyperplane theorem (henceforth LHT) states that up to homotopy equivalence X is obtained from Z by attaching cells of dimension n or larger, or equivalently that $\pi_i(X, Z) = 0$ for $i < n$. By the homological LHT we refer to the weaker statement that $H_i(X, Z) = H^i(X, Z) = 0$ for $i < n$.

The theorem fails for analytic spaces or complex varieties which are not smooth, although it still holds for projective varieties as long as the complement of the hyperplane section is a local complete intersection. See the book [GM88] for an exhaustive treatment of the classical LHT and its generalizations. In this note we go in a different direction and prove the theorem for smooth Deligne-Mumford stacks.

It is not surprising that with rational coefficients, the homological LHT holds for a variety with orbifold singularities. Poincaré-Lefschetz duality holds rationally for such varieties, so the fact that $X - Z$ is affine and thus $H_i(X - Z) = 0$ for $i > n$ implies the homological LHT. One of the main points of this note, then, is that the classical LHT still holds with *integral* coefficients (and in fact with homotopy groups) as long as one considers the homology of the stack instead of the underlying singular variety.

In Section 2 we show how the ideas of Bott's Morse theoretic proof of the LHT [Bot59] extend to the setting of DM stacks. We prove our main version of the LHT, which states that a stack can be constructed from a hyperplane section by a series of deformation retracts and attachments of high-dimensional disc quotients. Then we show how the usual statements

25 about relative homotopy groups and homology groups follow. The homotopy groups of a stack are defined to be the homotopy groups of its classifying space – this is discussed in more detail below.

In Section 3 we go back and carefully develop Morse theory of DM stacks. The main lemmas of Morse theory have been proven for the *underlying space* 30 of a DM stack in [Hep09a], but we observe that the proofs work for the stack itself. In this paper, a cell attachment refers to an honest 2-categorical colimit. This is the strongest notion of gluing discussed in [Noo05] – it doesn’t exist unless one is gluing along embedded substacks, and even then some care is required. Thus the main technical hurdle in Section 3 is showing 35 that a Morse function on \mathfrak{X} really does give a cell decomposition in the strongest sense.

Finally, in Section 4 we discuss the LHT for smooth Artin stacks. We prove a version of the homological LHT for differentiable stacks admitting presentations in which both X_1 and X_0 are compact (real) manifolds. We 40 also present some calculations which suggest that the homological LHT holds for a much larger class of Artin stacks.

Another important point in this note is a classical result on the stability of gradient descent flow. In Bott’s original paper, he assumes that the zero locus of s is “non-degenerate” in order to apply Morse theory. Here we prove 45 the LHT without any assumptions on the section. The key observation is that the gradient flow of a function is stable under much weaker hypotheses than the function being non-degenerate – as long as a function is real analytic (or close enough) its critical loci will have neighborhood deformation retracts. These ideas were developed in the work of Łojasiewicz[Łoj84], but 50 it’s a bit of real-analytic function theory that’s been overlooked in the Morse theory literature. We discuss some of Łojasiewicz’s results in Appendix A and refer to these ideas often throughout the paper.

The technical foundation for Morse theory of DM stacks has been laid by Hepworth in [Hep09a]. In that paper he develops Morse functions, Riemannian metrics, integration of vector fields, the strong topology on $C^\infty(\mathfrak{X})$, 55 and the main theorem of Morse theory for the underlying space $\tilde{\mathfrak{X}}$.¹ We will freely use these foundational constructions throughout this note without explicitly referencing them.

I’d like to thank my adviser, Constantin Teleman, for suggesting this 60 project to me and for many useful conversations throughout.

¹Although he doesn’t prove the main theorem on the level of stacks, which we do in Section 3

2. HYPERPLANE THEOREM FOR DM STACKS

Let \mathfrak{X} be an n -dimensional Deligne-Mumford stack² over the site of complex manifolds and \mathfrak{L} a holomorphic line bundle on \mathfrak{X} with hermitian structure \mathfrak{h} of type k (defined below). $s \in \Gamma(\mathfrak{X}, \mathfrak{L})$ will be a section of \mathfrak{L} , and $\mathfrak{Z} \subset \mathfrak{X}$ will denote the zero locus of s . We assume furthermore that $\bar{\mathfrak{X}}$, the coarse moduli space of \mathfrak{X} treated as a topological stack, is compact.

Choose an étale atlas $X_0 \rightarrow \mathfrak{X}$ such that $\mathfrak{L}|_{X_0}$ admits a nowhere vanishing holomorphic section σ . Then the $(1, 1)$ -form

$$\Theta_0 = \bar{\partial}\partial \log \mathfrak{h}(\sigma, \sigma) \in \Omega^{1,1}(X_0)$$

descends to a hermitian global section of $\Omega_{\mathfrak{X}}^{1,1}$, the curvature form Θ . In local analytic coordinates we can write Θ as $H_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$, and following [Bot59] we call $(\mathfrak{L}, \mathfrak{h})$ *type k* if the hermitian matrix $H_{\alpha\beta}$ is positive on a space of (complex) dimension k at every point.

Remark 2.1. This notion, although not mainstream, is the right one from the perspective of descent. $\Omega^{p,q}$ is a sheaf on the Artin site of complex manifolds, so the construction of Θ above works when $X_0 \rightarrow \mathfrak{X}$ is a surjective submersion and not necessarily étale. In addition, the property of being positive definite on a k -dimensional subspace is Artin-local, i.e. if $\varphi : X \rightarrow Y$ is a surjective submersion, then a hermitian form on Y is type k iff its pullback to X is type k . Thus we can define a type k holomorphic line bundle on an *Artin* stack.

Line bundles of type k arise naturally from maps to $\mathbb{C}\mathbb{P}^N$. Let $\varphi : \mathfrak{X} \rightarrow \mathbb{C}\mathbb{P}^N$ be a holomorphic map and $a : X_0 \rightarrow \mathfrak{X}$ an atlas, either étale or submersive. Define the minimal rank of φ to be the minimum over all $x \in X_0$ of the complex rank of $D_x(\varphi \circ a) : T_x X_0 \rightarrow T_{\varphi \circ a(x)} \mathbb{C}\mathbb{P}^N$. The minimal rank does not change if we pre-compose a with another surjective submersion $X'_0 \rightarrow X_0$, and thus the minimal rank of φ does not depend on the choice of atlas.

Choosing a hermitian form on \mathbb{C}^{n+1} induces a positive hermitian structure \mathfrak{h} on $\mathcal{O}_{\mathbb{C}\mathbb{P}^N}(1)$. If $\varphi : \mathfrak{X} \rightarrow \mathbb{C}\mathbb{P}^N$ has minimal rank k , then $\varphi^* \mathcal{O}(1)$ is type k on \mathfrak{X} , and a hyperplane $H \subset \mathbb{C}\mathbb{P}^N$ gives a section of $\varphi^* \mathcal{O}(1)$. In this case the LHT says that the relative homotopy groups $\pi_i(\mathfrak{X}, \varphi^{-1}(H))$, interpreted as the relative homotopy groups of the classifying spaces, vanish for $i < k$. Thus one can think of the generalization of the LHT to type k line bundles as a “large fiber” generalization as discussed in [GM88].

Now we will prove the LHT by analyzing the “Morse” function $f = \mathfrak{h}(s, s) \in C_{\mathbb{R}}^\infty(\mathfrak{X})$. Define the closed topological substack $\mathfrak{X}^{\leq \epsilon} = f^{-1}[0, \epsilon]$ and

²Meaning \mathfrak{X} admits a surjective representable étale map from a complex manifold $X_0 \rightarrow \mathfrak{X}$, and $\Delta : \mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ is representable and proper. Equivalently \mathfrak{X} admits a presentation by a proper étale Lie groupoid.

95 the open differentiable substack $\mathfrak{X}^\epsilon = f^{-1}[0, \epsilon)$. Note that $\mathfrak{X}^0 = f^{-1}(\{0\}) = \mathfrak{Z}$ is precisely the vanishing locus of s .

Fixing a Riemannian metric on \mathfrak{X} , we first show that f satisfies a Lojasiewicz inequality near its global minimum. This property of f guarantees that the gradient descent flow of f retracts \mathfrak{X}^ϵ onto \mathfrak{Z} for sufficiently small ϵ , but we will not make this precise until Section 3.

Claim 2.2. *For a sufficiently small $\epsilon > 0$, there are constants $C > 0$ and $\rho \in (0, 1)$ such that*

$$|f|^\rho \leq C|\nabla f| \quad (\text{L})$$

on the substack \mathfrak{X}^ϵ .

Proof. Note that it suffices to verify this inequality on $\bar{\mathfrak{X}}^\epsilon$. By compactness of $\bar{\mathfrak{Z}}$ it suffices to verify (L) around each $p \in \bar{\mathfrak{Z}}$. Choose an étale analytic coordinate patch $U \rightarrow \mathfrak{X}$ containing $p \in \bar{\mathfrak{Z}}$ such that $\mathfrak{L}|_U$ is trivial. Then $f_U = \mathfrak{h}(s, s) = h \cdot |s|^2$ is the product of a nonvanishing smooth function h with the real analytic function $|s|^2$, and thus by Theorem A.5 below the inequality (L) holds in a neighborhood of p . \square

Next we will perturb f to be Morse without affecting its values on $\mathfrak{X}^{\epsilon/2}$, so that our new function will continue to satisfy (L). Let $g : \mathfrak{X} \rightarrow [0, 1]$ be a smooth function such that $g|_{\mathfrak{X}^{\epsilon/2}} = 0$ and $g|_{\mathfrak{X} - \mathfrak{X}^\epsilon} = 1$.

Claim 2.3. *There is a small open neighborhood $U \subset C^\infty(\mathfrak{X})$ containing 0 such that for any $\eta \in U$, the new function $f' = f + g\eta$ has the properties*

- f' has no critical points on $\mathfrak{X}^\epsilon - \mathfrak{X}^{\epsilon/2}$
- f' is positive on $\mathfrak{X} - \mathfrak{X}^{\epsilon/2}$
- the hermitian form $\bar{\partial}\partial \log f'$ is positive on a space of dimension k at every point in $\mathfrak{X} - \mathfrak{X}^{\epsilon/2}$.

Proof. Multiplication by g induces a continuous map $C^\infty \mathfrak{X} \rightarrow C^\infty \mathfrak{X}$, so it suffices to prove that the set of η' for which $f + \eta'$ satisfies the above properties is open in $C^\infty \mathfrak{X}$ and contains 0.

120 Now choose a map $V \rightarrow \mathfrak{X}$ and an open subset $X_0 \subset V$ such that $X_0 \rightarrow \mathfrak{X}$ is surjective étale and $\text{cl}(X_0) \rightarrow \mathfrak{X}$ is proper. Then $C^\infty(\mathfrak{X})$ is identified as a vector space with invariant functions in $C^\infty(X_0)$, and one can show that for such an atlas the strong topology on $C^\infty(\mathfrak{X})$ is the subspace topology in $C^\infty(X_0)$. All three properties are open conditions on the derivatives of $(f + \eta)|_{X_0}$ up to order 2, and they are satisfied by f , so the claim follows. \square

Morse functions form a dense open subset of $C^\infty(\mathfrak{X})$, so there is some $\eta \in U$, where $U \subset C^\infty \mathfrak{X}$ is defined in Claim 2.3, such that $f + \eta$ is Morse. Consider the new function $f' = f + g\eta$. f' will satisfy all of the properties of Claim 2.3, and in addition it will be Morse on the substack $\mathfrak{X} - \mathfrak{Z}$, because f' contains no critical points in $\mathfrak{X}^\epsilon - \mathfrak{Z}$ and f' agrees with $f + \eta$ on $\mathfrak{X} - \mathfrak{X}^\epsilon$. From this point on we replace our function f with the new function f' .

Next we consider the index of the critical points of f away from \mathfrak{J} . Recall that for a critical point c , the tangent space $T_c\mathfrak{X}$ and the Hessian $H_c f$ are defined in the groupoid $U \times_{\mathfrak{X}} U \rightrightarrows U$, where $U \rightarrow \mathfrak{X}$ is an analytic coordinate patch containing c . The tangent space splits (non-canonically) as a representation of Aut_c , $T_c\mathfrak{X} \simeq T_c\mathfrak{X}_- \oplus T_c\mathfrak{X}_+$ on which $H_c f$ is negative definite and positive definite respectively. The index, ind_c is the isomorphism class of the representation $T_c\mathfrak{X}_-$ of Aut_c .

Of course in a local coordinate patch around c , the classical observation that the Hessian is negative definite on a space of dimension k still applies.

Claim 2.4. [Bot59] *At every critical point $c \in \bar{X} - \bar{Z}$, the index representation has $\dim(\text{ind}_c) \geq k$.*

Thus we have a function $f : \mathfrak{X} \rightarrow \mathbb{R}$ that is Morse on the open substack $\mathfrak{X} - \mathfrak{J}$ and such that $\mathfrak{J} \subset \mathfrak{X}$ is a *stable* global minimum of f . Now from the results of Morse theory in Section 3, we have the main theorem:

Theorem 2.5. *As a topological stack, \mathfrak{X} can be obtained from \mathfrak{J} by a finite sequence of deformation retracts and gluings of $[D^r/G]$ along $[S^{r-1}/G]$, where $r \geq k$. Here we can assume for each cell attachment that $G = \text{Aut}_c$ acting linearly on the unit disc in $T_c\mathfrak{X}_-$ for some $c \in \bar{\mathfrak{X}} - \bar{\mathfrak{J}}$.*

Remark 2.6. We only need \mathfrak{L} to be type k on the complement $\mathfrak{X} - \mathfrak{J}$. Geometrically, if \mathfrak{L} arises from a map $\phi : \mathfrak{X} \rightarrow \mathbb{C}\mathbb{P}^N$, this means we only need k to be the minimal rank of the restriction $\phi^{-1}(\mathbb{C}\mathbb{P}^N - H) \rightarrow \mathbb{C}\mathbb{P}^N - H$.

We will define more carefully what a deformation retract and a cell attachment are below, but suffice it so say that many corollaries about the various topological invariants follow.

Singular homology and cohomology are defined from the singular chain bi-complex of a simplicial manifold presenting \mathfrak{X} . The reader is referred to [Beh04] and [BX03] for an in-depth discussion and the equivalence between this and other notions of cohomology (i.e. de Rham or sheaf cohomology).

One thing that's not described in [Beh04] is that H^i and H_i are homotopy functors, but this follows from the usual argument, the key fact being the Poincaré lemma: by a local-to-global spectral sequence the projection $p : \mathfrak{X} \times I \rightarrow \mathfrak{X}$ induces an isomorphism in homology and cohomology.

The homotopy groups are a bit more subtle. We define $\pi_i(\mathfrak{X}) = \pi_i(\mathbb{B}\mathfrak{X})$ for a classifying space $\mathbb{B}\mathfrak{X}$. One way to define the classifying space $\mathbb{B}\mathfrak{X}$ is as the geometric realization of the simplicial manifold $X_p = X_0 \times_{\mathfrak{X}} \cdots \times_{\mathfrak{X}} X_0$ defined using an atlas $X_0 \rightarrow \mathfrak{X}$ – see [Seg68] for a definition and properties of this construction.

A more recent discussion using the modern language of stacks appears in [Noo08]. There the author uses the Haefliger-Milnor construction of a classifying space. In this note all stacks will admit presentations in which X_1 and X_0 are both metrizable – they are hoparacompact in the language of [Noo08] – in which case the Segal construction will suffice.

175 In any event, we will only need a few properties of $\mathbb{B}\mathfrak{X}$. First there is an epimorphism $\varphi : \mathbb{B}\mathfrak{X} \rightarrow \mathfrak{X}$ which is a universal weak equivalence, meaning that the base change along any map $T \rightarrow \mathfrak{X}$ is a weak equivalence. φ induces isomorphisms on homology and cohomology, in other words the homology of $\mathbb{B}\mathfrak{X}$ agrees with the double complex homology defined above.³

180 Also the construction $\mathbb{B}\mathfrak{X}$ is a *functor* from the category of (hoparacompact) stacks to the homotopy category of paracompact spaces. Finally, for $\mathfrak{Z} \subset \mathfrak{X}$ an embedding, we can construct $\mathbb{B}\mathfrak{Z}$ such that $\mathbb{B}\mathfrak{Z} \subset \mathbb{B}\mathfrak{X}$ is a subspace. Thus we define the relative homotopy groups $\pi_i(\mathfrak{X}, \mathfrak{Z}) := \pi_i(\mathbb{B}\mathfrak{X}, \mathbb{B}\mathfrak{Z})$. We omit base points from our notation, but the following theorem applies for
185 any base point in $\mathbb{B}\mathfrak{Z}$.

Corollary 2.7. *We have $\pi_i(\mathfrak{X}, \mathfrak{Z}) = 0$ for $i < k$. Hence from the long exact sequence of a pair $\pi_i(\mathfrak{Z}) \rightarrow \pi_i(\mathfrak{X})$ is an isomorphism for $i < k - 1$ and surjective for $i = k - 1$.*

Proof. For a triple of topological stacks $\mathfrak{Z} \subset \mathfrak{X} \subset \mathfrak{X}'$ we have a long exact sequence

$$\cdots \rightarrow \pi_i(\mathfrak{X}, \mathfrak{Z}) \rightarrow \pi_i(\mathfrak{X}', \mathfrak{Z}) \rightarrow \pi_i(\mathfrak{X}', \mathfrak{X}) \rightarrow \cdots$$

so if $\pi_i(\mathfrak{X}', \mathfrak{X}) = \pi_i(\mathfrak{X}, \mathfrak{Z}) = 0$, then $\pi_i(\mathfrak{X}', \mathfrak{Z}) = 0$ as well. Thus the corollary
190 follows by induction on the sequence of cell attachments and deformation retracts described in Theorem 2.5, once we verify that $\pi_i(\mathfrak{X}', \mathfrak{X}) = 0$ for $i < k$ when $\mathfrak{X} \subset \mathfrak{X}'$ is a disc attachment or deformation retract.

For a deformation retract, the inclusion $\mathbb{B}\mathfrak{X} \rightarrow \mathbb{B}\mathfrak{X}'$ is a homotopy equivalence, so $\pi_i(\mathfrak{X}', \mathfrak{X}) = 0$ for all i .

195 Next let $\mathfrak{X}' = \mathfrak{X} \cup_{[S/G]} [D/G]$ be a 2-categorical colimit, where D is a disc of dimension $\geq k$, S its boundary sphere, and G a finite group acting linearly on the pair (D, S) . Let $A \subset D$ be the complement of an open disc of half the radius of D , and consider the substack $\mathfrak{X} \cup_{[S/G]} [A/G] \subset \mathfrak{X}'$.

200 First observe that $A \rightarrow [A/G]$ and $D \rightarrow [D/G]$ are regular coverings, so $\pi_i([A/G]) \rightarrow \pi_i([D/G])$ is an isomorphism for $i < k - 1$ and surjective for $i = k - 1$; hence $([D/G], [A/G])$ is $k - 1$ -connected. The interiors of $\mathfrak{X} \cup_{[S/G]} [A/G]$ and $[D/G]$ cover \mathfrak{X}' , so excision for homotopy groups implies that $(\mathfrak{X}', \mathfrak{X} \cup_{[S/G]} [A/G])$ is $k - 1$ -connected as well.

205 Finally, the equivariant map $D \rightarrow D$ compressing A onto the boundary sphere S is equivariantly homotopic to the identity through a homotopy fixing S . Thus using the universal property of $\mathfrak{X} \cup_{[S/G]} [D/G]$ we have a homotopy equivalence of pairs $(\mathfrak{X}', \mathfrak{X} \cup_{[S/G]} [A/G]) \simeq (\mathfrak{X}', \mathfrak{X})$, so we deduce $\pi_i(\mathfrak{X}', \mathfrak{X}) = 0$ for $i < k$. \square

Recall that $H_*(\mathbb{B}\mathfrak{X}, \mathbb{B}\mathfrak{Z}) = H^*(\mathfrak{X}, \mathfrak{Z})$ and likewise for H^* , so

³If X_p is a simplicial presentation for \mathfrak{X} and X'_p the pulled-back presentation of $\mathbb{B}\mathfrak{X}$, then $X'_p \rightarrow X_p$ is a weak equivalence for all p , so the spectral sequence $E_1^{p,q} = H^q(X_p) \Rightarrow H^{p+q}(\mathfrak{X})$ implies that $H^*(\mathfrak{X}) \rightarrow H^*(\mathbb{B}\mathfrak{X})$ is an isomorphism.

210 **Corollary 2.8.** *For the relative homology and cohomology, with integer coefficients, we have $H^i(\mathfrak{X}, \mathfrak{Z}) = H_i(\mathfrak{X}, \mathfrak{Z}) = 0$ for $i < k$. As a consequence $H^i(\mathfrak{X}) \rightarrow H^i(\mathfrak{Z})$ is an isomorphism for $i < k - 1$ and injective for $i = k - 1$.*

2.1. **Topology of the underlying space.** For completeness we describe the consequences for the topology of the underlying spaces $\bar{\mathfrak{Z}} \subset \bar{\mathfrak{X}}$. In
 215 the introduction we mention that the homological LHT holds with rational coefficients for $\bar{\mathfrak{Z}} \subset \bar{\mathfrak{X}}$, but in fact we only need to invert the orders of the automorphism groups of points of \mathfrak{X} .

Proposition 2.9. *The underlying space $\bar{\mathfrak{X}}$ can be obtained from $\bar{\mathfrak{Z}}$, up to homotopy equivalence, by attaching finitely many disc quotients D^r/G along
 220 the boundary S^{r-1}/G . Here, as before, $r \geq k$ and $G = \text{Aut}_c$ acting linearly on the unit disc $D^r \subset T_c\mathfrak{X}_- \subset T_c\mathfrak{X}$.*

Using the Morse function f constructed above, we know that $\bar{\mathfrak{X}}^\epsilon$ deformation retracts onto $\bar{\mathfrak{Z}}$ for sufficiently small ϵ . From this point the proposition is a direct application of the Morse lemmas 7.5-7.7 in [Hep09a], using the
 225 estimate on the index of the critical points of f .

Corollary 2.10. *$H_i(\bar{\mathfrak{X}}, \bar{\mathfrak{Z}}; \mathbb{Z}[1/N]) = H^i(\bar{\mathfrak{X}}, \bar{\mathfrak{Z}}; \mathbb{Z}[1/N]) = 0$ for $i < k$ as long as N is a common multiple of the orders of the isotropy groups of \mathfrak{X} (of which there are finitely many because \mathfrak{X} is compact).*

Proof. This follows from the fact that $H_i(D/G, S/G; \mathbb{Z}[1/N]) = 0$ for $i <$
 230 k , which is really fact about group homology. One could also deduce the claim directly from Corollary 2.8 and the isomorphism $H_*(\mathfrak{X}; \mathbb{Z}[1/N]) \rightarrow H_*(\bar{\mathfrak{X}}; \mathbb{Z}[1/N])$. \square

3. MORSE THEORY OF DELIGNE-MUMFORD STACKS

Now we will prove the main theorem of the Morse theory for DM stacks
 235 used in the proof of Theorem 2.5. For this section we will take as input a function $f : \mathfrak{X} \rightarrow \mathbb{R}$ with the following properties:

- f is nonnegative with a global minimum $\mathfrak{Z} = f^{-1}\{0\} \subset \mathfrak{X}$ along which f satisfies a Lojasiewicz inequality (L).
- $\bar{f} : \bar{\mathfrak{X}} \rightarrow \mathbb{R}$ is proper
- 240 • f is Morse on the complement $\mathfrak{X} - \mathfrak{Z}$, with critical points c_1, c_2, \dots, c_r .

As with manifolds, the critical point data of f determines the “strong homotopy type” of \mathfrak{X} .

We fix a Riemannian metric g on \mathfrak{X} , and we will study the flow along the vector field $-\nabla f$.⁴

⁴On a DM stack a vector field corresponds to a consistent choice of vector field ξ_U on U for all étale $U \rightarrow \mathfrak{X}$. The tangent bundle of an Artin stack is more complicated [Hep09b]

Fact 3.1. Given a vector field ξ on \mathfrak{X} with compact support, there is a representable morphism $\Phi : \mathfrak{X} \times \mathbb{R} \rightarrow \mathfrak{X}$ such that for any étale map $M \rightarrow \mathfrak{X}$, the base change

$$\begin{array}{ccc} M' & \longrightarrow & \mathfrak{X} \times \mathbb{R} \\ \Phi' \downarrow & & \downarrow \Phi \\ M & \longrightarrow & \mathfrak{X} \end{array}$$

245 satisfies $D\Phi'(\frac{\partial}{\partial t})_{M'} = \xi_M$. Furthermore there is an isomorphism $e_\Phi : \Phi|_{\mathfrak{X} \times \{0\}} \Rightarrow \text{id}_{\mathfrak{X}}$, and the flow morphism Φ is uniquely determined by e_Φ .

The map Φ is group action of \mathbb{R} up to 2-isomorphism, and so it induces an \mathbb{R} action on the underlying space \mathfrak{X} and an \mathbb{R} action up to homotopy on the classifying space $\mathbb{B}\mathfrak{X}$.

250 We will use existence of flows to deconstruct the topology of \mathfrak{X} in three steps. First we show that $\mathfrak{X}^{\epsilon}]$ deformation retracts onto \mathfrak{Z} for sufficiently small ϵ . Then we show that if the interval $[a, b]$ contains no critical values of f , then $\mathfrak{X}^b]$ deformation retracts onto $\mathfrak{X}^a]$, and finally we show how the topology changes as a crosses a critical value.

255 First of all what does a deformation retract mean for stacks? Say $j : \mathfrak{Z} \rightarrow \mathfrak{X}$ is a closed immersion, then a *deformation retract* consists of a homotopy $h : \mathfrak{X} \times I \rightarrow \mathfrak{X}$ and a projection $\pi : \mathfrak{X} \rightarrow \mathfrak{Z}$, along with 2-morphisms

$$\begin{array}{ccc} \mathfrak{Z} \times I & \xrightarrow{j \times \text{id}} & \mathfrak{X} \times I \\ \text{pr}_1 \downarrow & \nearrow & \downarrow h \\ \mathfrak{Z} & \xrightarrow{j} & \mathfrak{X} \end{array} \qquad \begin{array}{ccc} \mathfrak{X} & \xrightarrow{h_0} & \mathfrak{X} \\ & \Downarrow & \\ & \text{id} & \\ \mathfrak{X} & \xrightarrow{h_1} & \mathfrak{X} \\ & \Downarrow & \\ & j \circ \pi & \end{array}$$

This definition is a categorification of the definition of deformation retract of spaces (or what's called a strong deformation retract by some authors), and implies that j is a homotopy equivalence. Both the geometric realization
260 functor and the underlying space functor respect products, so if \mathfrak{X} deformation retract onto \mathfrak{Z} in the above sense, then $\tilde{\mathfrak{X}}$ deformation retracts onto $\tilde{\mathfrak{Z}}$ and $\mathbb{B}\mathfrak{X}$ is homotopy equivalent to $\mathbb{B}\mathfrak{Z}$.

Remark 3.2. A strict deformation retract of topological groupoids induces
265 a deformation retract of stacks in the above sense.

Remark 3.3. A morphism factoring through a substack does so uniquely up to unique isomorphism, so the morphism π could be omitted from the definition of a deformation retract, instead just requiring that $h_1 : \mathfrak{X} \rightarrow \mathfrak{X}$ factor through $\mathfrak{Z} \hookrightarrow \mathfrak{X}$.

270 Thus if $h : \mathfrak{X} \times I \rightarrow \mathfrak{X}$ is a deformation retract of \mathfrak{X} onto \mathfrak{Z} , and $\mathfrak{X}' \subset \mathfrak{X}$ is a substack (not necessarily closed) containing \mathfrak{Z} such that $h|_{\mathfrak{X}' \times I} : \mathfrak{X}' \times I \rightarrow \mathfrak{X}$ factors through \mathfrak{X}' , then \mathfrak{X}' deformation retracts onto \mathfrak{Z} as well via this restricted h .

Remark 3.4. For an embedding of topological stacks $j : \mathfrak{Z} \rightarrow \mathfrak{X}$, we choose
 275 a presentation for \mathfrak{X} and form the mapping cylinder $\mathfrak{M}(j) = [M(j_1) \rightrightarrows M(j_0)] \subset \mathfrak{X} \times [0, 1]$. Morita equivalent presentations give Morita equivalent mapping cylinders because the mapping cylinder construction for spaces respects fiber products and epimorphisms. Furthermore, $\mathfrak{M}(j) \cong \mathfrak{X} \times \{0\} \cup_{\mathfrak{Z} \times \{0\}} \mathfrak{Z} \times [0, 1]$ is a pushout in the 2-category of topological stacks.
 280 Thus we can simplify our definition of a deformation retract further as a map $h : \mathfrak{X} \times I \rightarrow \mathfrak{X}$ such that h_1 factors through \mathfrak{Z} , along with an isomorphism $h|_{\mathfrak{M}(j)} \cong \text{pr}_1|_{\mathfrak{M}(j)}$.

We are now ready to prove the

Lemma 1. For $\epsilon > 0$ sufficiently small, \mathfrak{X}^ϵ contains no critical points of
 285 f , and the gradient descent flow of f provides a deformation retract of \mathfrak{X}^ϵ onto \mathfrak{Z} .

Proof. The key idea is to find, for ϵ small enough, an atlas for \mathfrak{X}^ϵ that is equivariant for the \mathbb{R} action. Start with an étale atlas $a : X_0 \rightarrow \mathfrak{X}$. $f|_{X_0}$ still satisfies the inequality (L) at every point in $Z_0 = (f_{X_0})^{-1}\{0\}$.

290 By Corollary A.4 there is an open subset $U_0 \subset X_0$ containing Z_0 such that $(-\nabla f)_{X_0}$ is integrable to a flow $\Phi_0 : U_0 \times [0, \infty) \rightarrow U_0$ which extends uniquely to a deformation retract $h_0 : U_0 \times [0, \infty) \rightarrow U_0$ onto Z_0 .

The étale map $U_0 \rightarrow \mathfrak{X}$ is equivariant with respect to the action of the semigroup $[0, \infty)$, so $[0, \infty)$ acts on $U_1 := U_0 \times_{\mathfrak{X}} U_0$ as well, and we would like to lift the action to ∞ , i.e. find a lift

$$\begin{array}{ccc}
 U_1 \times [0, \infty) & \xrightarrow{\Phi_1} & U_1 \\
 \downarrow & \dashrightarrow \exists! h_1 & \downarrow (s,t) \\
 U_1 \times [0, \infty) & \xrightarrow{(s,t)} & U_0 \times U_0 \times [0, \infty) \xrightarrow{h_0 \times h_0} U_0 \times U_0
 \end{array}$$

One shows the existence and uniqueness of h_1 using the fact that, because \mathfrak{X} is DM, $U_1 \rightarrow U_0 \times U_0$ is a proper immersion.

295 It suffices to show existence and uniqueness of h_1 in a neighborhood of each point $(\infty, p) \in [0, \infty) \times U_1$. If $(x, y) \in U_0 \times U_0$ is the image of (∞, p) , then there is a compact neighborhood N of (x, y) such that $(s, t)^{-1}(N)$ is a disjoint union of connected compact subsets, each mapped homeomorphically onto its image by (s, t) . Finally for a neighborhood of (∞, p) whose image in
 300 $U_0 \times U_0$ is contained in N there exists a unique lift.

Thus we get a map of groupoids $h : [0, \infty) \times (U_1 \rightrightarrows U_0) \rightarrow (U_1 \rightrightarrows U_0)$ providing a deformation retract of $U_1 \rightrightarrows U_0$ onto the subgroupoid $Z_1 \rightrightarrows Z_0$. By Remark 3.2 h , gives a deformation retract of the open substack $\mathfrak{U} = [U_1 \rightrightarrows U_0]$ onto \mathfrak{Z} . Because \mathfrak{Z} is compact, \mathfrak{X}^ϵ is contained in \mathfrak{U} for
 305 sufficiently small ϵ , so by Remark 3.3 the result follows. \square

Next we study what happens when there are no critical points between a and b .

Proposition 3.5. *If f has no critical points in $\mathfrak{X}^{[a,b]}$, then there is a deformation retract of $\mathfrak{X}^{[a,b]}$ onto \mathfrak{X}^a .*

Proof. Let ξ be a vector field with compact support such that $\xi \cdot f \leq 0$ and $\xi \cdot f = -1$ on $\mathfrak{X}^{[a,b]}$. Let $\Phi : \mathfrak{X} \times \mathbb{R} \rightarrow \mathfrak{X}$ be the flow of ξ and form the composition

$$h : \mathfrak{X}^{[a,b]} \times [0, 1] \xrightarrow{(\text{id}_{\mathfrak{X}^{[a,b]}}, \max(0, t(f-a)))} \mathfrak{X}^{[a,b]} \times [0, \infty) \xrightarrow{\Phi} \mathfrak{X}$$

310 \bar{f} decreases along the flow lines of $\bar{\Phi}$ and decreases with constant rate 1 in $\mathfrak{X}^{[a,b]}$, so it follows that $h : \mathfrak{X}^{[a,b]} \times [0, 1] \rightarrow \mathfrak{X}$ factors through $\mathfrak{X}^{[a,b]}$ and that h_1 factors through \mathfrak{X}^a .⁵ The isomorphism $e_\Phi : \Phi|_{\mathfrak{X} \times \{0\}} \Rightarrow \text{id}_{\mathfrak{X}}$ induces an isomorphism $h|_{\mathfrak{M}(j)} \simeq \text{pr}_1|_{\mathfrak{M}(j)}$ because $\max(0, t(f-a)) = 0$ on the substack $\mathfrak{M}(j) \subset \mathfrak{X}^{[a,b]} \times I$. \square

315 **Remark 3.6.** An argument identical to that in theorem 3.1 of [Mil63] or theorem 7.5 of [Hep09a] can also be used to show that $\mathfrak{X}^a \simeq \mathfrak{X}^{[a,b]}$ as stacks.

Proposition 3.7. *Let $c \in \mathbb{R}$ be a critical value of f , and assume that f is Morse near $f^{-1}\{c\}$. Denote the critical points in $f^{-1}\{c\}$ by p_1, \dots, p_r . Then for ϵ sufficiently small there is a closed topological substack*

$$\mathfrak{X}^{c-\epsilon] \subset \mathfrak{X}' \subset \mathfrak{X}^{c+\epsilon]}$$

such that $\mathfrak{X}^{c+\epsilon]}$ deformation retracts onto \mathfrak{X}' , and

$$\mathfrak{X}' \simeq \mathfrak{X}^{c-\epsilon] \cup \coprod_i [S_i / \text{Aut}_{p_i}] \coprod \coprod_i [D_i / \text{Aut}_{p_i}],$$

where D_i and S_i are the disc and sphere of radius ϵ in the index representation of Aut_{p_i} .

320 *Proof.* Again, the idea of the proof is identical to the discussion in [Mil63] or [Hep09a], one just has to check that the argument gives a cell attachment of stacks and not just underlying spaces.

For each p_i , identify an open substack containing p_i with $[U_{p_i} / \text{Aut}_{p_i}]$, where U_{p_i} is a ball around the origin in $T_{p_i}\mathfrak{X}$, and coordinates have been chosen in $T_{p_i}\mathfrak{X} = T_{p_i}\mathfrak{X}_+ \oplus T_{p_i}\mathfrak{X}_-$ such that the function $f|_{U_i} = c + |u_+|^2 - |u_-|^2$. Choose an ϵ smaller than the radii of all of these Morse coordinate patches, and define the closed substack of $\mathfrak{X}^{c+\epsilon]}$

$$\mathfrak{X}' = \{f \leq c - \epsilon\} \cup \bigcup_i \{u_+ = 0 \text{ and } |u_-|^2 \leq \epsilon\}$$

Where we have used the slightly informal notation $\{u_+ = 0 \text{ and } |u_-|^2 \leq \epsilon\}$ for the closed substack $[D_i / \text{Aut}_{p_i}]$.

325 Theorem 16.9 of [Noo05] gives a criterion for checking that \mathfrak{X}' is in fact a 2-categorical union. First of all, \mathfrak{X}' is manifestly a gluing in the language of [ibid.], in that $\mathfrak{X}' \setminus \coprod_i [S_i / \text{Aut}_{p_i}] \simeq \mathfrak{X}^{c-\epsilon] \sqcup \coprod_i [(D_i \setminus S_i) / \text{Aut}_{p_i}]$. The

⁵A map $\psi : \mathfrak{Y} \rightarrow \mathfrak{X}$ factors through $\mathfrak{X}^{[a,b]}$ if and only if $f \circ \psi : \mathfrak{Y} \rightarrow \mathbb{R}$ factors through $(-\infty, b]$, so it suffices to check on the level of underlying spaces.

second criterion – the existence of an atlas $a : X'_0 \rightarrow \mathfrak{X}'$ such that the pair of invariant subspaces $a^{-1}([S_i/\text{Aut}_{p_i}]) \subset a^{-1}([D_i/\text{Aut}_{p_i}])$ satisfies the local left lifting property (LLLP) with respect to the class of covering maps – is tautologically satisfied because every pair satisfies the LLLP with respect to local homeomorphisms.⁶ So we have really verified that a union of two substacks of a DM stack is always a 2-categorical union.

The rest of the argument for proceeds exactly as in [Mil63] and [Hep09a]. One introduces an auxiliary function F such that: F agrees with f outside of the Morse coordinate patches (on which $F \leq f$), $\{F \leq c + \epsilon\} = \mathfrak{X}^{c+\epsilon}$, and $F(p_i) < c - \epsilon$. Then one uses Proposition 3.5 to retract $\mathfrak{X}^{c+\epsilon}$ onto $\{F \leq c - \epsilon\}$, a substack which differs from $\mathfrak{X}^{c-\epsilon}$ only within the Morse coordinate patches. Finally one constructs a manifestly Aut_{p_i} equivariant deformation retract of $\{F \leq c - \epsilon\}$ onto \mathfrak{X}' in each $[U_i/\text{Aut}_{p_i}]$ which restricts to the identity on $\mathfrak{X}^{c-\epsilon} \cap [U_i/\text{Aut}_{p_i}]$. These can then be glued to give the global deformation retract of $\{F \leq c - \epsilon\}$ onto \mathfrak{X}' . \square

For reference we now summarize the main theorem of Morse theory for compact Deligne-Mumford stacks which we have just proven

Theorem 3.8. *Let $f : \mathfrak{X} \rightarrow \mathbb{R}$ be a smooth nonnegative function on a compact Deligne-Mumford stack \mathfrak{X} achieving its global minimum on the closed topological substack $f^{-1}\{0\}$. Assume that*

- *f satisfies a Lojasiewicz inequality (L) at every $p \in \bar{\mathfrak{Z}}$, and*
- *f is Morse on the open substack $\mathfrak{X} - \bar{\mathfrak{Z}}$.*

Then \mathfrak{X} can be obtained from $\bar{\mathfrak{Z}}$ by a finite sequence of deformation retracts and 2-categorical attachments of $[D_i/\text{Aut}_{p_i}]$ along $[S_i/\text{Aut}_{p_i}]$ where p_i ranges over the critical points of f in $\bar{\mathfrak{X}} - \bar{\mathfrak{Z}}$ and $D_i \subset T_{p_i}\bar{\mathfrak{X}}$ is a small disc with boundary S_i .

4. HOMOLOGICAL LEFSCHETZ THEOREM FOR ARTIN STACKS

The LHT for Artin⁷ stacks is beyond the current reach of Morse theoretic techniques. The theory of integration of vector fields on Artin stacks has been worked out in [Hep09b], but Morse theory has not been developed yet.

In this section we describe a local-to-global proof of the homological LHT that applies to differentiable Artin stacks admitting a presentation by compact manifolds.

In the complex category, proper Artin stacks are close to being gerbes, so in order to have a version of the LHT with broader applications, we will leave the category of complex stacks. For instance, the following theorem applies to a global quotient of a compact complex manifold by a compact Lie group.

⁶Although it is not necessary here, one could even find an atlas $a : X'_0 \rightarrow \mathfrak{X}'$ for which $a^{-1}([D_i/\text{Aut}_{p_i}]) = D_i$ and likewise for S_i

⁷An Artin stack is just a differentiable stack admitting a surjective submersion from a manifold

365 **Theorem 4.1.** *Let \mathfrak{L} be a hermitian line bundle on a differentiable Artin stack \mathfrak{X} , and let s be a section. Assume that there exists a proper submersion $X_0 \rightarrow \mathfrak{X}$ from a compact complex manifold X_0 such that the pullback $L_0 = \mathfrak{L}|_{X_0}$ is holomorphic of type k and the section $s|_{X_0}$ is holomorphic.*

370 *As before let $\mathfrak{Z} \subset \mathfrak{X}$ be the topological substack on which s vanishes. Then $H^i(\mathfrak{X}, \mathfrak{Z}) = H_i(\mathfrak{X}, \mathfrak{Z}) = 0$ for $i < k$.*

Proof. From the proper submersion $X_0 \rightarrow \mathfrak{X}$ we get a closed, full, saturated sub-groupoid $Z_\bullet \subset X_\bullet$ presenting $\mathfrak{Z} \subset \mathfrak{X}$. Let $\{X_p\}$ and $\{Z_p\}$ denote the respective simplicial nerves.

375 We can form the smooth function $f = |s|^2$ on \mathfrak{X} , and as before \mathfrak{Z} is precisely the zero locus of f . In particular Z_p is the zero locus of $f|_{X_p} = (f|_{X_0})|_{X_p}$. Additionally Z_p is the preimage of Z_0 under any of the $p + 1$ simplicial maps $X_p \rightarrow X_0$.

380 Now the function $f|_{X_0}$ is the norm squared of a section of a hermitian line bundle of type k . Thus as in Section 2 above we can perturb $f|_{X_0}$ to a function which we call ϕ_0 which

- satisfies a Lojasiewicz inequality near its global minimum Z_0 ,
- is Morse(-Bott) away from Z_0 with critical points of index $\geq k$.

385 Define the functions $\phi_p = \phi_0|_{X_p}$ using the 0^{th} simplicial face map $X_p \rightarrow X_0$ to restrict to X_p . Note that ϕ_0 is no longer invariant (does not necessarily descend to a smooth function on \mathfrak{X}), so we must be explicit about which simplicial maps we use. Both bulleted properties are preserved when one restricts along a surjective submersion, so the ϕ_p are functions with Lojasiewicz global minima $Z_p = \phi_p^{-1}\{0\}$ that are Morse-Bott away from Z_p with critical manifolds of index $\geq k$. It follows from non-degenerate Morse theory that

390 $H_i(X_p, Z_p) = H^i(X_p, Z_p) = 0$ for all $i < k$.

From the bicomplex computing $H_*(\mathfrak{X}, \mathfrak{Z})$ we have a homological spectral sequence

$$E_{p,q}^1 = H_q(X_p, Z_p) \Rightarrow H_{p+q}(\mathfrak{X}, \mathfrak{Z}) \quad (1)$$

so the vanishing of $H_i(X_p, Z_p)$ implies $H_i(\mathfrak{X}, \mathfrak{Z}) = 0$ for $i < k$. A similar spectral sequence implies the result for cohomology. \square

395 **Example 4.2.** This theorem applies to global quotients even when the group does not act holomorphically. For instance let $G = \mathbb{Z}/2\mathbb{Z}$ acting on $\mathbb{C}\mathbb{P}^l$ by complex conjugation $[z_0 : \cdots : z_l] \mapsto [\bar{z}_0 : \cdots : \bar{z}_l]$. By identifying $\mathbb{C}^{l+1} = \mathbb{C} \otimes \mathbb{R}^{l+1}$, real subspaces of \mathbb{R}^{l+1} induce complex subspaces of \mathbb{C}^{l+1} , and thus we have an embedding $\mathbb{R}\mathbb{P}^l \subset \mathbb{C}\mathbb{P}^l$ as the fixed locus of G .

400 Complex conjugation acts equivariantly on $\mathcal{O}_{\mathbb{C}\mathbb{P}^l}(1)$, and an invariant holomorphic section corresponds to a form with real coefficients $s = r_0 z_0 + \cdots + r_l z_l$. The zero locus of s is a hypersurface $H \subset \mathbb{C}\mathbb{P}^l$ induced by a real hypersurface in \mathbb{R}^{l+1} , and by a real coordinate change we may assume that $H = \{[0 : * : \cdots : *]\}$.

Let $p = [1 : 0 : \cdots : 0] \in \mathbb{C}\mathbb{P}^l$. Then $\mathbb{C}\mathbb{P}^l - \{p\}$ deformation retracts equivariantly onto H , so we have

$$\begin{aligned} H_G^i(\mathbb{C}\mathbb{P}^l, H) &= H_G^i(\mathbb{C}\mathbb{P}^l, \mathbb{C}\mathbb{P}^l - \{p\}) = H_G^i(\mathbb{C}^l, \mathbb{C}^l - \{0\}) \\ &\simeq H^{i-l}(\mathbb{B}[*/\mathbb{Z}/2]; \Omega) \end{aligned}$$

405 In this last expression, we have used the Thom isomorphism for the bundle $[\mathbb{C}^l/G] \rightarrow [*/G]$, and Ω is a local system on $\mathbb{B}[*/G]$ which depends on the orientability of this bundle. Thus we have $H_G^i(\mathbb{C}\mathbb{P}^l, H) = 0$ for $i < l$.

Example 4.3. Non-holomorphic group actions also arise naturally as subgroups of the group of unit quaternions acting by left multiplication on $\mathbb{P}(\mathbb{H}^l)$, where choose either i, j , or k as the complex structure on \mathbb{H}^l .
410

Example 4.4. Let X be a projective variety of dimension n and $\mathfrak{X} \rightarrow X$ a gerbe for a compact group G , and let $Z \subset X$ be a hyperplane section. The theorem implies that the cohomology of \mathfrak{X} agrees with the cohomology of the restricted gerbe $\mathfrak{Z} \rightarrow Z$ in degree $< n - 1$.

415 In this case we can also see this by comparing the Leray-Serre spectral sequence $E_2^{p,q} = H^p(X; H^q(BG)) \Rightarrow H^{p+q}(\mathfrak{X})$ and the corresponding sequence for $\mathfrak{Z} \rightarrow Z$. The result follows because, by the classical LHT, the restriction map on the E^2 page $H^p(X; H^q(BG)) \rightarrow H^p(Z; H^q(BG))$ is an isomorphism for $p < n - 1$.

420 Unlike Theorem 4.1, the Leray-Serre argument in the example above does not require G to be compact, which suggests that Theorem 4.1 might hold for a much larger class of Artin stacks. We also observe that the homological LHT also applies to global quotients by reductive groups, giving further evidence for a more general statement than Theorem 4.1.

425 **Corollary 4.5.** *Let G be a compact Lie group and let the complexification $G_{\mathbb{C}}$ act on a compact complex manifold X , and let $(\mathfrak{L}, \mathfrak{h})$ be a $G_{\mathbb{C}}$ -equivariant hermitian line bundle of type k with invariant global section s vanishing on $Z \subset X$, then $H_i([X/G_{\mathbb{C}}], [Z/G_{\mathbb{C}}]) = H^i([X/G_{\mathbb{C}}], [Z/G_{\mathbb{C}}]) = 0$ for $i < k$.*

Proof. $G_{\mathbb{C}}$ deformation retracts onto G , and so the pair $(X \times (G_{\mathbb{C}})^{\times p}, Z \times (G_{\mathbb{C}})^{\times p})$ deformation retracts onto $(X \times (G)^{\times p}, Z \times (G)^{\times p})$ for any number of factors p . Thus by the spectral sequence (1), $H_i([X/G], [Z/G]) \rightarrow H_i([X/G_{\mathbb{C}}], [Z/G_{\mathbb{C}}])$ is an isomorphism, and $H_i([X/G], [Z/G]) = 0$ for $i < k$ by Theorem 4.1. \square
430

APPENDIX A. STABILITY OF GRADIENT DESCENT FLOW

435 We have used Morse-theoretic techniques to prove the LHT without requiring the section of the line bundle to be non-degenerate as in [Bot59]. The global statements in the text above rely on the local results summarized in this appendix. Here we prove the key fact: that the gradient descent flow of a function near its global minimum is *stable* as long as it satisfies the
440 following property

Definition A.1. Let X be a Riemannian manifold and f a smooth function on X . We say that f satisfies a *Lojasiewicz inequality* at a point p if there are constants $\rho \in (0, 1)$ and $C > 0$ such that

$$|f - f(p)|^\rho \leq C|\nabla f| \quad (\text{L})$$

in some neighborhood of p

As a consequence of (L), f takes the value $f(p)$ at any critical point near p . In particular if Z is the global minimal set of f and (L) holds at every point of Z , then Z is isolated from the rest of the critical locus of f .

445 **Proposition A.2.** *If f achieves its global minimum along $Z \subset X$ and (L) holds at $p \in Z$, then there is an open set $U \subset X$ containing p such that*

(1) *the flow $\phi_t(x)$ of $-\nabla f$ is defined for all $x \in U$ and all $t \geq 0$, and $\phi_t(x) \in U$.*

450 (2) *The map $U \times [0, \infty) \rightarrow U$ given by $(x, t) \mapsto \phi_t(x)$ extends uniquely to a continuous map $U \times [0, \infty] \rightarrow U$. In particular the flow of $-\nabla f$ deformation retracts U onto $U \cap Z$.*

Remark A.3. The proposition is equally valid for a local minimum. Also, the argument below shows that any gradient like vector field for f will have the same stability property.

455 *Proof.* For simplicity we will assume that $f \geq 0$ and $f(p) = 0$. Let $\phi_t(x_0)$ be an integral curve starting at $x_0 \in X$. As long as $\phi_t(x_0)$ stays in a region in which (L) holds, an arc length integral gives a uniform bound

$$\text{dist}(x_0, \phi_t(x_0)) \leq \frac{C}{1-\rho} f(x_0)^{1-\rho} \quad \text{for } t \geq 0 \quad (2)$$

Start with a relatively compact open ball $B_{r'}$ of radius r' around p on which the inequality (L) holds. Now we can find a smaller ball $B_r \subset B_{r'}$ 460 such that the difference in radii $r' - r$ is larger than $Cf(x_0)^{1-\rho}/(1-\rho)$ for any point $x_0 \in B_r$. It follows from (2) that any flow line starting in B_r stays within $B_{r'}$ under the (positive time) flow of $-\nabla f$, and thus by the escape lemma [Lee03] the flow is defined on B_r for all $t \geq 0$. We let $U = \bigcup_{t \geq 0} \phi_t(B_r)$.

465 Inequality (L) implies that $f(\phi_t(x_0)) \rightarrow 0$ as $t \rightarrow \infty$. Combining this with (2) shows that $\phi_t(x_0)$ is Cauchy as $t \rightarrow \infty$ and remains in a compact region, so $\phi_\infty(x) = \lim_{t \rightarrow \infty} \phi_t(x)$ is a well defined function on U . In fact (2) shows that $(x, t) \mapsto \phi_t(x)$ is uniformly continuous as $t \rightarrow \infty$, and so the gradient descent flow extends uniquely to a continuous map $U \times [0, \infty] \rightarrow U$. 470 Finally, $\phi_s(\phi_\infty(x)) = \lim_{t \rightarrow \infty} \phi_{s+t}(x) = \phi_\infty(x)$, so $\phi_\infty(U) \subset Z$ and the map constructed above is a deformation retract of U onto $U \cap Z$. \square

An immediate corollary of Proposition A.2 is the global statement that

Corollary A.4. *Let f be a smooth nonnegative function on X with global
475 minimum $Z = f^{-1}\{0\} \subset X$. If the inequality (L) holds at each $p \in Z$, then
there is an open neighborhood U of Z on which the negative gradient flow
extends uniquely to a map $U \times [0, \infty] \rightarrow U$ which is a deformation retract
of U onto Z .*

In addition if Z is compact then X^{ϵ} deformation retracts onto Z for
480 sufficiently small ϵ .

We have shown above that the Lojasiewicz inequality implies a stable
gradient flow, but we are left with the question of which functions satisfy
(L). First of all, functions satisfying (L) have the following properties.

(1) if h is a smooth function with $h(p) \neq 0$ and f satisfies (L) with
485 $f(p) = 0$, then so does hf

(2) Let f and g satisfy (L) with $f(p) = g(p) = 0$. If the angle between
 ∇f and ∇g is bounded away from π near p then $f + g$ satisfies (L),⁸
and if the angle between ∇f^2 and ∇g^2 is bounded away from π then
490 fg satisfies (L).

(3) if $\pi : X \rightarrow Y$ is an open mapping near $p \in X$ and $f \circ \pi$ satisfies (L)
at p then f satisfies (L) at $\pi(p) \in Y$. If π is submersive at p , then
the converse is true as well. Letting π be the identity map shows
495 that (L) is independent of the metric.

(4) if $\pi : X \rightarrow Y$ is proper and surjective onto a neighborhood of $q \in Y$
and $f \circ \pi$ satisfies (L) at every point in the fiber $\pi^{-1}\{q\}$, then f
satisfies (L) at q .

500 One might wonder in light of property (2) above if arbitrary algebraic
combinations of functions will still satisfy (L), but this turns out to be
false. For instance if $\rho : \mathbb{R} \rightarrow \mathbb{R}$ vanishes to all orders at 0, then $f(x) =$
 $(1 + \rho)x - x = \rho x$ does not satisfy (L) at 0 even though $(1 + \rho)x$ and x
do. Thus while inequality (L) has convenient dynamical properties, it lacks
505 algebraic properties. It is thus surprising that many functions do satisfy the
Lojasiewicz inequality:

Theorem A.5. *Let f be smooth function on X which in some choice of
local coordinates is a non-vanishing smooth function times a real analytic
function which vanishes at p , then f satisfies property (L) at p .*

510 *Proof.* Using real analytic resolution of singularities there is a proper sur-
jective analytic map $\pi : Y \rightarrow X$ such that $f \circ \pi = h(y)y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ where
 y_1, \dots, y_n are local analytic coordinates on Y and h is a smooth function
which does not vanish near $\pi^{-1}\{p\}$. Applying property (2) above, or by
a direct calculation, one shows that the monomial $y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ satisfies (L)

⁸More precisely the condition is $(\nabla f, \nabla g) \geq (-1 + \epsilon)|\nabla f||\nabla g|$ in a neighborhood of p
for some small ϵ

515 along its zero locus, and thus by property (1) so does $f \circ \pi$. Finally by
property (4) f satisfies (L) at p . \square

Lojasiewicz was the first to prove that real analytic functions satisfy the
inequality (L) and apply it to the topology of real-analytic varieties. Many
authors have extended his work, for instance in [KMP00].

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