Using Lagrange Multipliers In Math 53

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1 The one-constraint problem, and the method

Suppose you want to maximize (or minimize) a differentiable function $f(\overline{x}) = f(x, y, z)$ (actually any number of variables will work) subject to the constraint $g(\overline{x}) = c$ where c, is a constant. This means you are only considering inputs $\overline{x} = (x, y, z)$ satisfying the equation $g = c$, and you want to know which such inputs, if any, maximize the function f, and what its maximum value is.

Suppose also that our constraint equation is *nonsingular*. This means that at all points of the set $\{g = c\}$, the function g is differentiable and $\nabla g \neq 0$. Then:

Theorem 1.1 (Lagrange Multipliers Theorem 1) If $\overline{x}$ is such a maximizing (or minimizing) input, then $\overline{x}$ also satisfies the equation $\nabla f = \lambda \nabla g$ for some scalar λ. In 3 coordinates:

$$\begin{aligned} g &= c \\ f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \end{aligned}$$

In other words, you can solve this system of equations to find a list of f-inputs to check, then list their f-outputs, and one of these **"check points"** will be the maximum if it exists.

Algebra tips:

1) **Solving for λ and equating the results** is often a useful way to start, athough sometimes a clever trick will work faster.

2) **Dividing by 0 will lead you down the path of darkness!** It is *essential* when you solve this (or any!) system of equations that, when you divide by or "cancel off" an expression, you consider a *separate case* where that expression equals 0. This is because $AB = AC$ implies that $B = C$ *or* $A = 0$ (or both). You will miss check points if you do this wrong!

3) **Finding "too many" check points** is not a problem, as long as they all satisfy the constraint $g = c$: the output checking process will eliminate points not yielding the max or min. The important thing is not to miss any!

Presentation and sanity check tips:

1) **Make a list of (and/or draw a box around) all your "check points"** at the end of the problem, i.e. all the f-inputs you checked, along with their outputs. This allows both you and the grader to see in summary what you have done.

2) **Indicate clearly which cases you have considered** to be sure you get a good grade, like by underlining them. It also doesn't hurt if they're organized!

2 An awesome example:

Find the maximum and minimum of $x^2 + y^3$ given the (nonsingular) constraint $x^4 + y^6 = 2$.

Solution: Let $f(x, y) = x^2 + y^3$ and $g(x, y) = x^4 + y^6$, and write the Lagrange system of equations:

$$\begin{array}{lrcl} (1) & x^4 + y^6 & = & 2 \\ (2) & 2x & = & \lambda 4x^3 \\ (3) & 3y^2 & = & \lambda 6y^5 \end{array}$$

We want to divide by x and by y to solve for λ, so we must consider the (two!) cases where x or y is 0:

- Case $x = 0$: Then (1) says $y^6 = 2$ and $y = \pm\sqrt[6]{2}$, so we found the check points $\boxed{(0, \pm\sqrt[6]{2})}$.
- Case $y = 0$: Then (1) says $x^4 = 2$ so $x = \pm\sqrt[4]{2}$, so we found $\boxed{(\pm\sqrt[4]{2}, 0)}$.
- Now we can assume both $x, y \neq 0$, and divide by x and y to obtain $\lambda = 1/(2x^2) = 1/(2y^3)$, so $x^2 = y^3$. Then (1) says $x^4 + x^4 = 2$ so $x = \pm 1$, and $y^3 = x^2 = 1$, so we found $\boxed{(\pm 1, 1)}$.

(*Finally, we check the outputs of the points we found, and indicate which yield the max/min, and we're done!*)

$$\boxed{\begin{array}{rcll} f(0, \sqrt[6]{2}) & = & \sqrt{2} & \\ f(0, -\sqrt[6]{2}) & = & -\sqrt{2} & \leftarrow \text{MIN} \\ f(\pm\sqrt[4]{2}, 0) & = & \sqrt{2} & \\ f(\pm 1, 1) & = & 2 & \leftarrow \text{MAX} \end{array}}$$

Remarks / why this example is so awesome:

1) Notice that without considering (separately!) the (separate!) cases $x = 0, y = 0$, we would never have found the minimum!

2) If in the third case when $x^2 = 1$, we found y by solving $x^4 + y^6 = 2$, we'd get extraneous check points $(\pm 1, -1)$. These don't satisfy the full Lagrange system (because $x^2 \neq y^3$), but that doesn't matter: since, they satisfy the constraint equation, the checking process is guaranteed to rule them out! Indeed, $f(\pm 1, -1) = 0$ is neither a max nor a min. **The moral of the story:** extraneous checkpoints are okay as long as they satisfy the original constraint, and you check them in the end.

3) $\mathbf{x = 0, y = 0, z = 0}$ **and** $\boldsymbol{\lambda = 0}$ **are not teh only separate cases evar!** If you divide by or cancel $x - 1$, then $x = 1$ is a separate case! If you divide by or cancel $y - z^2$, then $y = z^2$ is a separate case!

3 The two-constraint problem, and the method

Suppose instead you are maximizing f with *two constraints*, $g = c$ and $h = d$ where c, d are constants and g and h are functions of $\overline{x} = (x, y, z)$ (or any number of at least 3 variables).

Suppose also that our constraints are *nonsingular*. This means that at all points of the set $\{g = c,\ h = d\}$, the functions g,h are differentiable and $\nabla g, \nabla h$ are not parallel (and hence not 0 either). Then:

Theorem 3.1 (Lagrange Multipliers Theorem 2) If $\overline{x}$ is such a maximizing (or minimizing) input, then $\overline{x}$ also satisfies the equation $\nabla f = \lambda \nabla g + \mu \nabla h$ for some scalars λ, μ. In 3 coordinates:

$$\begin{aligned} g &= c \\ h &= d \\ f_x &= \lambda g_x + \mu h_x \\ f_y &= \lambda g_y + \mu h_y \\ f_z &= \lambda g_z + \mu h_z \end{aligned}$$

In other words, just like before, you can solve this system of equations to find a list of f-inputs to check, then list their f-outputs, and one of these **"check points"** will be the maximum if it exists.

These theorems are quite geometric in meaning, and it is easy to understand intuitively *why* they are true (see my notes explaining them). For answers to common questions about less favorable circumstances, see the next page.

4 Stuff you should be curious but not worried about because Math 53 is easier than real life

Should I solve the constraint?

- Don't worry! In math 53, I will try not to give you problems where your best option is to solve the constraint!
- In real life, sometimes instead of using the constraint for Lagrange multipliers, you can solve it for one variable and substitute into f to get a new optimization problem with fewer variables and no constraint. This is sometimes easier, but not always.

What if no maximum exists?

- Don't worry! In math 53, if I ask you to find it, you can assume that it exists!
- In real life, if the constraint set $\{g = c\}$ is *bounded* (continuity of g ensures the set is closed) and f is continuous, then a maximum must exist.
- If the constraint set $\{g = c\}$ is *unbounded*, then limit techniques can sometimes be used to check whether there is a maximum.

What if f is not differentiable?

- Don't worry! In math 53, I will usually give you a differentiable function! (If not, I will somehow hint that there may be a problem.)
- In real life, you should consider separately any inputs where ∇f is discontinuous or undefined (since f is differentiable away from these points). This just means you have more "check points".

What if the constraint is singular?

- Don't worry! In math 53, I will always give you non-singular constraints!
- In real life, you should consider separately any inputs where ∇g is discontinuous, undefined, or 0. If there is a second constraint $h = d$, you should also consider separately inputs where ∇h is discontinuous, undefined, or $\nabla h = \nu \nabla g$, i.e. they are parallel. (The constraints are nonsingular away from these points). This just means you have more "check points".
- Sometimes singular constraints can be replaced by nonsingular ones. For example, the constraint $(x - y)^2 = 0$, which is singular, can be replaced by $x - y = 0$, which is not.