

Curves and Stuff

Key: [M V.4] =Miranda, Algebraic Curves and Riemann Surfaces chap. V sect. 4  
 (Rough Outline)

- Weil = formal sum of codimension one [Hartshorne II.6]
- $X$  smooth proj. curve over  $\mathbb{C}$ ,  $\text{Pic}(X) \leftrightarrow \text{Cl}(X)$ , constructing a map to projective space from global sections of  $\mathcal{O}(D)$  [<http://math.stanford.edu/~vakil/725> class 21-23]
- Linear systems, basepoints, gdr's, embeddings, very ample [M V.4]
- The degree of a smooth proj. curve [M V.2] ← also discusses canonical embedding
- Riemann-Roch [Hartshorne IV.1, M VI.3]
- (didn't get to it in the talk but go learn it!) Cuves of low genus [M VII.1, VII.2]

Some highlights

Set up:  $X =$  compact Riemann Surface = complex projective curve,  $\mathcal{M}$  sheaf of meromorphic functions,  $D = \sum_{p \in X} n_p \cdot p$  a divisor. Given  $f \in \mathcal{M}(X)$ , can form divisor  $(f) = \sum_{p \in X} \text{ord}_p(f) \cdot p$ .  
 Fact:  $\deg(f) = \sum_p \text{ord}_p(f) = 0$ . From  $D$  can form invertible sheaf  $\mathcal{O}(D)$  defined as

$$\mathcal{O}(D)(U) = \{f \in \mathcal{M}(U) \mid (f) + D \geq 0\} \text{ (i.e. the divisor } (f) + D \text{ is effective)}$$

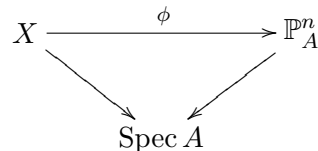
Set  $L(D) = \Gamma(X, \mathcal{O}(D))$ . **If**  $\deg D < 0$  **then**  $L(D) = 0$ . For if  $f \in L(D)$  is nonzero then  $(f) + D$  is effective so has degree  $\geq 0$ , but  $\deg((f) + D) = \deg D$  by comments above, which contradicts  $\deg D < 0$ . Same idea can be used to show that if  $\deg D = 0$ , then  $L(D)$  is only nonzero when  $\mathcal{O}(D) \cong \mathcal{O}_X$ .

$\{(f) + D \mid f \in L(D)\} = |D| =$  set of all effective divisors linearly equivalent to  $D$  is the complete linear series (or system) of  $D$ , any linear subspace of  $|D|$  is a linear series. A base point is a point where some  $f$  has a pole, or all  $f$ 's have a zero. For curves, you can always get rid of base points and  $\{ \text{Base pt. free linear sys. of dim } n \} \leftrightarrow \{ \phi: X \hookrightarrow \mathbb{P}^n \}$

For  $\phi_D: X \hookrightarrow \mathbb{P}^n$  the map defined by  $n + 1$  sections  $s_0, \dots, s_n \in \Gamma(X, \mathcal{O}(D)) := L(D)$ , we have  $\phi_D$  is an embedding if for every  $p, q \in X$  (including  $p = q$ )

$$\dim L(D - p - q) = \dim L(D) - 2$$

More General Hartshorne Statement (didn't really discuss in the talk) Let  $A$  be a commutative ring,  $X$  a scheme over  $A$ , and  $\mathbb{P}_A^n = \text{Proj } A[x_0, \dots, x_n]$ . If we have



Then  $\phi^*(\mathcal{O}(1))$  is invertible sheaf on  $X$ , generated by  $\phi^*(x_i)$ . Conversely, if  $L$  is invertible sheaf on  $X$  generated by global sections then there is an  $A$ -morphism  $\psi: X \rightarrow \mathbb{P}_A^n$  such that  $L \cong \psi^*(\mathcal{O}(1))$ . In this case  $X$  has a cover by  $X_i = \{p \in X \mid (s_i)_p \notin m_p L_p\}$  where  $s_0, \dots, s_n$  are global sections that generate  $L$ .

General criterion for  $\phi: X \rightarrow \mathbb{P}_A^n$  to be an embedding (Prop. 7.2): The opens  $X_i$  are affine and for the map  $X_i \rightarrow U_i \subset \mathbb{P}_A^n$ , is defined by a surjective ring homomorphism  $A[y_0, \dots, \hat{y}_i, \dots, y_n] \rightarrow \Gamma(X_i, \mathcal{O}_{X_i})$