Curves and Stuff
Key: [**M V.4**] =Miranda, Algebraic Curves and Riemann Surfaces chap. V sect. 4
(Rough Outline)

- Weil = formal sum of codimension one [Hartshorne II.6]
- X smooth proj. curve over $\mathbb{C}$, $\mathrm{Pic}(X) \leftrightarrow \mathrm{Cl}(X)$, constructing a map to projective space from global sections of $\mathcal{O}(D)$ [`http://math.stanford.edu/~vakil/725` class 21-23]
- Linear systems, basepoints, gdr's, embeddings, very ample [**M V.4**]
- The degree of a smooth proj. curve [**M V.2**] ← also discusses canonical embedding
- Riemann-Roch [Hartshorne IV.1, **M VI.3**]
- (didn't get to it in the talk but go learn it!) Cuves of low genus [**M VII.1, VII.2**]

<u>Some highlights</u>
Set up: X = compact Riemann Surface = complex projective curve, $\mathcal{M}$ sheaf of meromorphic functions, $D = \sum_{p\in X} n_p \cdot p$ a divisor. Given $f \in \mathcal{M}(X)$, can form divisor $(f) = \sum_{p\in X} ord_p(f) \cdot p$. Fact: $\deg(f) = \sum_p ord_p(f) = 0$. From D can form invertible sheaf $\mathcal{O}(D)$ defined as

$$\mathcal{O}(D)(U) = \{f \in \mathcal{M}(U) | (f) + D \geq 0\} \text{(i.e. the divisor } (f) + D \text{ is effective)}$$

Set $L(D) = \Gamma(X, \mathcal{O}(D))$. **If** $\deg D < 0$ **then** $L(D) = 0$. For if $f \in L(D)$ is nonzero then $(f) + D$ is effective so has degree ≥ 0, but $\deg((f) + D) = \deg D$ by comments above, which contradicts $\deg D < 0$. Same idea can be used to show that if $\deg D = 0$, then $L(D)$ is only nonzero when $\mathcal{O}(D) \cong \mathcal{O}_X$.

$\{(f) + D | f \in L(D)\} = |D|$ = set of all effective divisors linearly equivalent to D is the complete linear series (or system) of D, any linear subspace of $|D|$ is a linear series. A bases point is a point where some f has a pole, or all f's have a zero. For curves, you can always get rid of base points and { Base pt. free linear sys. of dim n } $\leftrightarrow \{\phi\colon X \hookrightarrow \mathbb{P}^n\}$

For $\phi_D\colon X \hookrightarrow \mathbb{P}^n$ the map defined by $n + 1$ sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{O}(D)) := L(D)$, we have ϕ_D is an embedding if for every $p, q \in X$ (including $p = q$)

$$\dim L(D - p - q) = \dim L(D) - 2$$

More General Hartshorne Statement (didn't really discuss in the talk) Let A be a commutative ring, X a scheme over A, and $\mathbb{P}^n_A = \mathrm{Proj}\, A[x_0, \ldots, x_n]$. If we have

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & \mathbb{P}^n_A \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}\, A & \end{array}$$

Then $\phi^*(\mathcal{O}(1))$ is invertible sheaf on X, generated by $\phi^*(x_i)$. Conversely, if L is invertible sheaf on L generated by global sections then there is an A-morphism $\psi\colon X \to \mathbb{P}^n_A$ such that $L \cong \psi^*(\mathcal{O}(1))$. In this case X has a cover by $X_i = \{p \in X | (s_i)_p \notin m_p L_p\}$ where $s_0, \ldots, s_n$ are global sections that generate L.

General criterion for $\phi\colon X \to \mathbb{P}^n_A$ to be an embedding (Prop. 7.2): The opens X_i are affine and for the map $X_i \to U_i \subset \mathbb{P}^n_A$, is defined by a surjective ring homomorphism $A[y_0, \ldots, \hat{y}_i, \ldots, y_n] \to \Gamma(X_i, \mathcal{O}_{X_i})$