A sequence of functions which is uniformly convergent on compact sets but not locally uniformly convergent

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Suppose $f_n : X \to Y$ is a sequence (or a net) of functions from a topological space X into a uniform space Y (e.g. a metric space). Then f_n is *locally uniformly convergent* if every point of X has a neighborhood V such that f_n is uniformly convergent on V, and f_n is *compactly* convergent if f_n is uniformly convergent on every compact subset $K \subseteq X$.

Local uniform convergence implies compact convergence, and the converse is true when X is locally compact. One might expect the converse to fail in general, because how could the proof possibly work without local compactness?

Proposition. *Compact convergence* $\not\Rightarrow$ *local uniform convergence.*

The challenge here is to come up with a space X which is not locally compact, but which has a simple characterization of its compact subsets that gives us flexibility in designing compactly convergent sequences, and then hope to make one which is not locally uniformly convergent.

As a set, let $X = \mathbb{R}^2 \cup \{p\}$, the plane with an extra point, and inside X let $Y = \{\text{the } y\text{-axis}\} \cup \{p\}$. Declare that each usual open set $U \subseteq \mathbb{R}^2$ is open in X, and for each usual open set $V \subseteq \mathbb{R}^2$ which contains the y-axis, declare that $V \cup \{p\}$ is open in X. This declaration of open sets is stable under finite intersections and arbitrary unions and is thus a topology on X.

Any neighborhood N of p contains an open set $W \supseteq Y$, which contains a smaller open set $W' \supseteq Y$ obtained by shrinking the x-coordinates of $W \cap \mathbb{R}^2$ by a factor of 1/2. If N were compact, then the closed subset $N \setminus W' \subseteq$

$\mathbb{R}^2$ would be compact, but it contains the unbounded set $W \setminus W' \subseteq \mathbb{R}^2$, a contradiction. Hence X is not locally compact at p, and so at least it has a chance!

Claim. *If $K \subseteq X$ is compact, then $K \setminus Y$ is bounded.*

Proof. Suppose $K \subseteq X$ is compact, so in particular, every infinite sequence in K must have a cluster point (a point every neighborhood of which contains infinitely many values of the sequence).

Extend the coordinate function x to X by letting $x(p) = 0$. Then x is continuous, so the x-coordinates of K are bounded. Now suppose $K \setminus Y$ is unbounded, so it must have unbounded y-coordinates, w.l.o.g. unbounded above. Choose $q_n \in K \setminus Y$ such that $y(q_n)$ strictly increases to infinity. Let $m_n = \min(|x(q_1)|, |x(q_2)|, \ldots, |x(q_n)|)$, and note that $m_n > 0$.

Let $V_n = \{(a,b) \in \mathbb{R}^2 \, : \, a < y(q_n) \, , \, |b| < m_n\}$. Then $V = \bigcup_i V_i$ is a neighborhood of the y-axis which excludes every q_n, and thus $V \cup \{p\}$ is a neighborhood of p which excludes every q_n. Hence $\{q_n\}$ does not cluster at p, and clearly does not cluster at any point of $\mathbb{R}^2$. This is a contradiction. □

Corollary. Every compact set $K \subseteq X$ is contained in a set of the form $C \cup Y$ where $C \subseteq \mathbb{R}^2$ is compact.

Proof. Let C be the closure of $K \setminus Y$ in $\mathbb{R}^2$, which is bounded by the claim above and hence compact. □

Proof of proposition. Define $f_n : X \to \mathbb{R}$ as follows:

$$f_n(q) = \begin{cases} 0 & \text{if } q \in Y \\ 0 & \text{if } q \in X \setminus Y \text{ and } |y(q)| < n \\ 1 & \text{if } q \in X \setminus Y \text{ and } |y(q)| \geq n \end{cases}$$

Then $\{f_n\}$ is constant at 0 on Y, and is eventually 0 on any compact $C \in \mathbb{R}^2$, hence converges uniformly to 0 on any compact subset K of X by the corollary above. However, if W is any neighborhood of p, then $W \setminus Y$ contains points of arbitrarily large y-value, so $\sup_W(f_n) = 1$. Hence $\{f_n\}$ cannot converge uniformly to 0 on any open $W \ni p$, so it is not locally uniformly convergent at p. □

Even though the original problem was solved, I couldn't resist classifying all the compact subsets of X:

Proposition. *A subset of X is compact if and only if is of the form C or $A \cup C\backslash Y$, where $C \subseteq \mathbb{R}^2$ is compact and A is an arbitrary subset of Y containing p.*

Proof. Any such $A \cup C\backslash Y$ is compact since an open cover of it $\{U_i\}$ involves at least one neighborhood of p, say U_1, which contains all of Y and thus A, and $\{U_{i\neq 1}\}$ reduces to a finite cover of the compact set $C\backslash U_1$.

Conversely, suppose $K \subseteq X$ is compact. If $K \subseteq \mathbb{R}^2$ then take $C = K$ and we are done. Otherwise, $p \in K$. Take $A = K \cap Y$ which must contain p, and let C be the closure of $K\backslash Y$ in $\mathbb{R}^2$, which is compact because $K\backslash Y$ is bounded. Clearly $C\backslash Y \supseteq K\backslash Y$, so we get $A \cup C\backslash Y \supseteq A \cup K\backslash Y = K$, and for the reverse inclusion it suffices to show that $C\backslash Y \subseteq K\backslash Y$.

We know that $C \subseteq \overline{K}$, so $C\backslash Y \subseteq \overline{K}\backslash Y$. Now, the intersection of K with any closed disk in $X\backslash Y \subseteq \mathbb{R}^2$ is a closed subset of K, hence compact, hence closed in $\mathbb{R}^2$. In other words, K is locally closed in $X\backslash Y$, i.e. $\overline{K}\backslash Y = K\backslash Y$ and we are done. $\square$