My Qualifying Exam
(or, How I Spent My Summer Vacation)
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My committee members were Bjorn Poonen, Don Sarason, Paul Vojta, and Eyvind Wichmann (physics). I'll give the questions and correct answers (some of which I did not come up with during the actual exam!).

Topic 1: Algebraic Number Theory

Question: Is 73 a square (mod 415)? Mod 83? **[Vojta]**

Answer: 73 is not a square (mod 415), despite the fact that the Jacobi symbol $(\frac{73}{415}) = 1$, because it's not even a square mod 5. An easy application of quadratic reciprocity shows that 73 is not a square mod 83.

Question: What are the squares in $\mathbb{Q}_2$? **[Poonen]**

Answer: The squares are 0 and the elements of the set

$$2^{2\mathbb{Z}}(1 + 8\mathbb{Z}_2).$$

To see this, it is enough to show that if $\nu_2(x) = 0$ or 1, then x is a square in $\mathbb{Q}_2$ iff $x \equiv 1 \pmod 8$. But this claim follows immediately from the "strong" version of Hensel's Lemma proved in Lang's book, for instance.

Question: Which primes of $\mathbb{Q}$ split/stay inert/ramify in $\mathbb{Q}(\sqrt{5})$? **[Vojta]**

Answer: As is easy to verify, the ring of integers is $\mathbb{Z}[\alpha]$, where $\alpha = \frac{1+\sqrt{5}}{2}$ has minimal polynomial $x^2 - x - 1$. In this situation, we need only check the behavior of this polynomial mod p to determine how (p) factors. So: 2 is inert, 5 ramifies, and for $p \neq 2, 5$, we have: p splits iff 5 is a square mod p and p is inert otherwise.

Question: Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and let E be the compositum of all degree-10 extensions of $\mathbb{Q}$ inside $\overline{\mathbb{Q}}$. Show that $E \neq \overline{\mathbb{Q}}$. **[Poonen]**

Answer: The simplest thing to do is to note that no element of prime degree larger than 10 can be in E (an easy exercise). However, you could also embed everything in $\mathbb{Q}_p$ and use Krasner's lemma a couple of times. I confess I don't fully understand this method.

Topic 2: Algebraic Geometry

Question: What is a proper morphism? Is the morphism $\mathbb{A}^1_k \to \operatorname{Spec} k$ proper? (Assume k is algebraically closed.) What is the valuative criterion of properness? Give a (nontrivial) example of a proper morphism. **[Vojta]**

Answer: A proper morphism $f : X \to Y$ is one that is separated (the diagonal map $\Delta : X \to X \times_Y X$ is a closed immersion), finite type (there is a cover of Y by open affines $V_i \cong \operatorname{Spec} B_i$ such that $f^{-1}(V_i)$ can be covered by open affines $U_{ij} \cong \operatorname{Spec} A_{ij}$ such that A_{ij} is a finite B_i-algebra), and universally closed (if Y' is a Y-scheme, then $f' : X \times_Y Y' \to Y'$ sends closed sets to closed sets).

The map $\mathbb{A}^1_k \to \operatorname{Spec} k$ cannot be proper because it is not universally closed: the base-extended map $\mathbb{A}^2_k \to \mathbb{A}^1_k$ is not closed, as the hyperbola $xy = 1$ projects down to the complement of a point in $\mathbb{A}^1$.

The valuative criterion of properness is in Hartshorne's book; my TEX skills are too limited for me to reproduce the commutative diagram needed in its full beauty.

Any projective morphism of noetherian schemes is proper, so for instance the map $\mathbb{P}^1_k \to \operatorname{Spec} k$ is proper.

Question: Let X be a variety over k (I think k should probably be algebraically closed), let Y be a closed affine subscheme of X, and let Z be a closed subscheme of X which is proper over k. Then show that $Y \cap Z$ is a finite scheme over k. [**Poonen**]

Answer: First recall that $Y \cap Z$ is given a scheme structure via the product construction: $Y \cap Z = Y \times_X Z$. Since $Y \cong \operatorname{Spec} A$ is affine and the morphism $Y \times_X Z \to Y$ is a closed immersion (because closed immersions are stable under base extension), $Y \cap Z$ is also affine, isomorphic to $\operatorname{Spec}(A/I)$ for some ideal I. Because X is finite type, Y is too; this implies that A is a finitely generated k-algebra, and therefore so is A/I. We wish to show that A/I is a finite k-vector space.

Well, the map $Y \cap Z \to Z \to \operatorname{Spec} k$ is the composition of two proper maps (closed immersions are proper), and so it's proper itself. Thus we have a proper scheme over k, and a theorem of Serre (I believe) says that $\Gamma(Y \cap Z, O_{Y \cap Z})$ is a finite k-vector space; of course this is just A/I since $Y \cap Z$ is affine.

Question: Let A be a ring and M a finitely generated A-module. Let $x_1, \ldots, x_n$ be elements of M whose images span $M/\mathfrak{m}M$ for every maximal ideal $\mathfrak{m}$. What can you say about M? What does this have to do with algebraic geometry? [**Vojta**]

Answer: You can actually say that $x_1, \ldots, x_n$ generate M as an A-module. To see this, let N be the A-module generated by the x_i and form an exact sequence

$$0 \to N \to M \to P \to 0.$$

Tensoring with $A/\mathfrak{m}$, we get

$$N/\mathfrak{m}N \to M/\mathfrak{m}M \to P/\mathfrak{m}P \to 0,$$

but the first map is an isomorphism by the assumption, so $P/\mathfrak{m}P = 0$ for all $\mathfrak{m}$.

Localizing commutes with dividing by $\mathfrak{m}$, so $P_\mathfrak{m}/\mathfrak{m}P_\mathfrak{m} = 0$ as well. Now $P_\mathfrak{m}$ is an $A_\mathfrak{m}$-module, so Nakayama's lemma gives $P_\mathfrak{m} = 0$ (note we need that P is finitely generated here, which follows from M being finitely generated). Now this holds for all $\mathfrak{m}$. So let x be an element of P. Since the image of x is zero in all the $P_\mathfrak{m}$, for all $\mathfrak{m}$ there exists $a \notin \mathfrak{m}$ such that $ax = 0$. Thus the annihilator of x is an ideal not contained in any maximal ideal. Thus it is all of A, so $x = 0$. Thus $P = 0$ and $N = M$.

In algebraic geometry, this means that if X is an affine scheme, then a necessary and sufficient condition that a quasi-coherent O_X-module be coherent is that it is generated by global sections. (Exercise.) I'm sure there are other ways to translate this as well...

Topic 3: Complex Analysis

Question: Explain Weierstrass products. [**Wichmann**]

Answer: Suppose we are given an infinite sequence of complex numbers $\{z_i\}$ such that $\lim z_n = \infty$. (Assume for simplicity's sake that none of the z_n are zero; no real generality is lost.) Then we can create an entire function with simple zeroes precisely at the z_n, and the most general function of this type is of the form

$$e^{g(z)} \Pi_{n=1}^{\infty} \epsilon_{k_n}(\frac{z}{z_n}),$$

where $g(z)$ is entire, k_n is chosen sufficiently large, and

$$\epsilon_k(y) = (1-y)\exp(\sum_{j=1}^{k} \frac{y^j}{j}).$$

The point is that you'd like to multiply all the $(1 - \frac{z}{z_n})$, but this might not converge; to nudge the factors of the infinite product back towards 1, one introduces these truncated exponentials. Note that as k grows, $\epsilon_k(y)$ begins to look like $(1-y)\exp(-\log(1-y)) = 1$. So if k_n is large enough, the product converges. (Ahlfors remarks that $k_n = n$ suffices, but in practice k_n is chosen much smaller.)

Question: Compute $\int_{-\infty}^{\infty} e^{-x^2}\cos(x)\,dx$. **[Sarason]**

Answer: Let γ_R be the rectangular contour oriented counterclockwise with vertices at the points $\pm R$ and $\pm R + i/2$. Then note that as $R \to \infty$, the integral of e^{iz-z^2} along the vertical parts of the contour goes to 0. By Cauchy's theorem, we get

$$0 = \lim_{R\to\infty} \int_{\gamma_R} e^{iz-z^2}\,dz = \int_{-\infty}^{\infty} e^{ix-x^2}\,dx - \int_{-\infty}^{\infty} e^{i(x+i/2)-(x+i/2)^2}\,dx,$$

so the integral we want is the real part of

$$\int_{-\infty}^{\infty} e^{-1/4}e^{-x^2}\,dx = e^{-1/4}\sqrt{\pi},$$

which is real itself, so that is our answer.