Patrick Corn
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Solution Set 9

Problem 1: Let $f(z) = \prod_{i=1}^{n}(z - a_i)$. Then f is a polynomial, and so it's holomorphic everywhere. Note that $|f(0)| = 1$, so since the maximum of $|f|$ over the unit disk is attained on the boundary, there must be a point p with $|p| = 1$ such that $|f(p)| \geq 1$. And this is what we wanted.

Problem 2: Let R be a large enough real number, i.e. large enough that all the roots of q are enclosed by the circle C_R. Note that

$$\left| \int_{C_R} \frac{p(z)}{q(z)}\, dz \right| \leq 2\pi R M_R$$

where M_R is the maximum value of $\left|\frac{p(z)}{q(z)}\right|$ on C_R. Now let $n = \deg(q) - \deg(p)$, and note that $\lim_{z\to\infty} \frac{z^n p(z)}{q(z)}$ is a finite number. So $\limsup_{R\to\infty} R^n M_R$ is finite. Since $n > 1$, it follows that $\lim_{R\to\infty} RM_R = 0$, so that

$$\lim_{R\to\infty} \int_{C_R} \frac{p(z)}{q(z)}\, dz = 0$$

But

$$\int_{C_R} \frac{p(z)}{q(z)}\, dz = \int_{C} \frac{p(z)}{q(z)}\, dz$$

for large enough R, by Cauchy's theorem for an annulus. So then $\int_C \frac{p(z)}{q(z)}\, dz = 0$.

Problem 3: (a) The polynomial in the denominator has four distinct roots, equal to $e^{2\pi ik/5}$ for $1 \leq k \leq 4$. These are all simple poles. Also, it is not hard to see that ∞ is a simple pole:

$$f\left(\frac{1}{z}\right) = \frac{1}{z} \cdot \frac{1}{z^4 + z^3 + z^2 + z + 1},$$

and the latter fraction is holomorphic near 0.

(b) It is not hard to see that the zeroes of $\sin z$ are precisely the integer multiples of π. They are all simple zeroes, because the derivative of $\sin z$ doesn't vanish at any of them. Therefore $z = k\pi$ is a pole of order 2 (a double pole) for all $k \in \mathbb{Z}$.

The case of ∞ is more interesting. Be careful–in fact ∞ is not an isolated singularity at all! There is no punctured disk around ∞ contained in the region G on which $\frac{1}{\sin^2 z}$ is holomorphic, because $k\pi \notin G$ for all $k \in \mathbb{Z}$.

(c) First of all, ∞ is a removable singularity, because $\sin z$ is holomorphic at 0. The only other possible problem is at $z = 0$. We can write the Laurent series for $\sin \frac{1}{z}$ in powers of $\frac{1}{z}$:

$$\sin\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{-2n-1}}{(2n+1)!}$$

so 0 must be an essential singularity.

Problem 4: For the finite set of points $F = \{z_1, \ldots, z_n\}$, we can find a nonnegative integer p_k, $1 \leq k \leq n$, such that $(z - z_k)^{p_k} f(z)$ can be extended to a holomorphic function in a neighborhood of z_k (because each z_k is either removable, so $p_k = 0$, or a pole of order p_k). Then consider

$$g(z) = f(z) \prod_{k=1}^{n} (z - z_k)^{p_k}$$

Then $g(z)$ is an entire function, and by assumption it has a non-essential singularity at ∞. As we have seen in class, this implies that $g(z)$ is a polynomial. (It is equal to its Taylor series centered at 0, which can only have finitely many nonzero terms.) Thus f is a rational function.

Problem 5: We can write $h(z) = (z - z_0)h_1(z)$ with h_1 holomorphic in a neighborhood of z_0, and $h_1(z_0) \neq 0$. Then

$$\frac{g(z)}{h(z)} = (z - z_0)^{-1} \frac{g(z)}{h_1(z)}$$

and g/h_1 is holomorphic in a neighborhood of z_0, hence representable by a Taylor series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ there. Then the coefficient of $(z - z_0)^{-1}$ in the Laurent series for g/h is equal to $a_0 = \frac{g(z_0)}{h_1(z_0)}$. But, as we have seen many times already, $h_1(z_0) = h'(z_0)$.

Problem 6: (a) There are q singularities of the form $\zeta_k = e^{2\pi ik/q}$, $0 \leq k \leq q - 1$. By the previous exercise, the residue at ζ_k is equal to $\frac{z^p}{-qz^{q-1}}$ evaluated at ζ_k. This equals

$$\frac{\zeta_k^p}{-q\zeta_k^{q-1}} = -\frac{\zeta_k^{p+1}}{q} = -\frac{e^{2\pi ik(p+1)/q}}{q}$$

(b) There are double poles at ±1, so we cannot use the previous exercise. Consider the function $f_1(z) = z^5/(z+1)^2$. It has a Taylor series $\sum_{n=0}^{\infty} a_n(z-1)^n$, and the Laurent series of our function around 1 is equal to this Taylor series divided by $(z-1)^2$. Then the coefficient of $(z-1)^{-1}$ in that Laurent series will be equal to a_1. This equals $f_1'(1)$, and now we must do a little computation to get $f_1'(1) = 1$, so the residue at $z = 1$ is 1.

Similarly let $f_{-1}(z) = z^5/(z-1)^2$. The residue at $z = -1$ will equal $f_{-1}'(-1) = 1$. So both residues are 1.

(c) There are two simple poles, ζ and ζ^2, where $\zeta = e^{2\pi i/3}$. We can use the previous exercise to find that the residue at ζ^k is the function $\frac{\cos z}{2z+1}$ evaluated at ζ^k. Let's just leave it like that:

$$\operatorname{res}_{z=\zeta^k} \frac{\cos z}{1+z+z^2} = \frac{\cos \zeta^k}{2\zeta^k + 1}, \quad k = 1, 2.$$

(d) There are simple poles at $z = k\pi$, $k \in \mathbb{Z}$, so we can use the above exercise to see that

$$\operatorname{res}_{z=k\pi} \frac{1}{\sin z} = \frac{1}{\cos k\pi} = (-1)^k.$$

Problem 7: Let $f(z) = (z - z_0)^m f_1(z)$, where $f_1(z)$ is holomorphic in a neighborhood of z_0 and $f_1(z_0) \neq 0$. Then

$$\frac{f'(z)}{f(z)} = \frac{m(z-z_0)^{m-1}f_1(z) + (z-z_0)^m f_1'(z)}{(z-z_0)^m f_1(z)} = \frac{m}{z-z_0} + \frac{f_1'(z)}{f_1(z)}$$

and the latter fraction is a holomorphic function in a neighborhood of z_0, so the principal part of the Laurent series of f'/f is exactly $\frac{m}{z-z_0}$. So the residue is m, as desired.

Problem 8: First, if $g(z)$ is identically zero, there's nothing to prove. Otherwise, the zeroes of $g(z)$ are isolated, as we've seen, so $h(z) = f(z)/g(z)$ is holomorphic everywhere except for isolated singularities. Let z_0 be one such singularity; we see immediately that

$$\limsup_{z \to z_0} |h(z)| < \infty$$

since $|h(z)| \leq 1$ on a punctured neighborhood of z_0. Thus z_0 is a removable singularity. So all the (non-∞) singularities of $h(z)$ are removable. So $h(z)$ is actually an entire function. But it's bounded! So by Liouville's theorem it is constant.