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Solution Set 8

Problem 1: Suppose such an f exists. Let $g(z) = z^2$; then f and g agree on a set with a limit point, namely the set $\{\frac{1}{2k} \colon k \in \mathbb{N}\}$, and so they must agree everywhere by the identity theorem, which is absurd.

Problem 2: We've done exercise II.8.2. It shows that the function $g(z) = \overline{f(\overline{z})}$ is holomorphic on $\{\overline{z} \colon z \in G\}$. In this case, that set equals G. At any rate, f and g are holomorphic functions that, by assumption, agree on $G \cap \mathbb{R}$. Clearly $G \cap \mathbb{R}$ is open in $\mathbb{R}$, and if it were empty, then $G \cap H_+$ and $G \cap H_-$ would be disjoint open sets whose union was G; since G is symmetric with respect to the real axis, they would both have to be nonempty, which would contradict the connectedness of G. So $G \cap \mathbb{R}$ contains some open interval in $\mathbb{R}$, which certainly is a set with a limit point. Thus $f = g$ on G, which gives us what we want.

Problem 3: Suppose that $|f|$ attains a local minimum in G at z_0, but $f(z_0) \neq 0$. Then find a neighborhood N of z_0 in G such that $|f(z)| \geq |f(z_0)|$ for all $z \in N$. Of course f cannot vanish on N, so the function $g(z) = 1/f(z)$ is holomorphic on N, and it has a local maximum at $z_0 \in N$, which implies, by the maximum modulus principle, that g, and hence f, is constant on all of N. Then f is constant on all of G, by the identity theorem (it agrees with the zero function on an open disk N, which certainly has lots of limit points). So there you go.

Problem 4: Fix $w \in D$, where D is the open unit disk. Then define

$$\phi_1(z) = \frac{z - w}{1 - z\overline{w}}, \quad \phi_2(z) = \frac{z - f(w)}{1 - z\overline{f(w)}}$$

Both ϕ_1 and ϕ_2 are LFTs sending D to D (you can see that the unit circle maps to itself by multiplying the denominator by $\overline{z}$ and taking absolute values; see also exercise III.9.4 and III.9.5). Now then, the function $g = \phi_2 \circ f \circ \phi_1^{-1}$ is a holomorphic map from D to itself, and $g(0) = 0$. Then by Schwarz's lemma, we have $|g(s)| \leq |s|$ for all $s \in D$. Set $s = \phi_1(z)$; then we get

$$|\phi_2(f(z))| \leq |\phi_1(z)|$$

which is precisely the statement of Pick's lemma. (Note that the part about $|g'(0)| \leq 1$ is exactly what you need to do exercise VII.17.3, which I did not assign.)

Lastly, equality holds in Schwarz's lemma if and only if $g(z) = cz$, and in this case $f = \phi_2^{-1} \circ g \circ \phi_1$ is a composition of three LFTs, so it's an LFT sending D to itself. Also note that if f is an LFT sending D to itself, then g is an LFT sending D to itself with $g(0) = 0$; it is clear that the only possibility for g is that it is a dilation $g(z) = cz$. (Again, see exercise III.9.4 or III.9.5.) So equality holds if and only if f is an LFT sending D to itself.

Problem 5: There are a couple of ways to do this; you can use the result of the previous exercise, or think about adapting its proof. I think I'll do the latter. Suppose that z and w are two distinct fixed points. Then define ϕ_1 and ϕ_2 as above. But, of course, in this situation $\phi_1 = \phi_2$. At any rate, we see that $g = \phi_1 \circ f \circ \phi_1^{-1}$ is a holomorphic function from D to itself, and $f(w) = w$ implies that $g(0) = 0$, just as in the last problem. Note also that $g(\phi_1(z)) = \phi_1(f(z)) = \phi_1(z)$. We can apply Schwarz's lemma to see that $|g(s)| \leq |s|$ for all $s \in D$, and equality actually holds for $s = \phi_1(z)$. Therefore $g(z) = cz$ for some constant c. But $g(z) = cz$ cannot have a nonzero fixed point, unless $c = 1$. In this case, $f = \phi_1^{-1} \circ g \circ \phi$ is also the identity function, because g is.

Problem 6: Suppose u is a positive harmonic function on $\mathbb{C}$. We can find a holomorphic function $g(z)$ with real part equal to u. Let ϕ be an LFT sending the right half plane to the open unit disk, for instance $\phi(z) = \frac{z-1}{z+1}$. Then $\phi \circ g$ is an entire function, because ϕ is holomorphic on the image of g. And the image of $\phi \circ g$ is contained in the open unit disk. So we have ourselves a bounded entire function, which must be identically equal to some constant c by Liouville's theorem. Then g is also constant, equal to $\phi^{-1}(c)$. So $u = \operatorname{Re} g$ is constant.

Problem 7: Ok, well, there's an obvious problem here: K could be very small, like a finite set of points, in which case the statement fails miserably. So what I need to assume is that K has nonempty interior. So let's assume that from now on.

So suppose f has no zeroes on K, and $|f|$ is constant on ∂K. Then we can apply the maximum modulus principle to f and $1/f$, to see that the maximum absolute value on K of f and of $1/f$ is attained on the boundary. So the maximum and minimum of $|f|$ over K is attained on its boundary. But $|f|$ is constant on the boundary, so the maximum and minimum of $|f|$ over K are the same; in other words, $|f|$ is constant on K.

Now take an open disk N contained in K (this is where we are using that K has nonempty interior). On N, f must be a constant. (This is another old exercise, II.8.1(c).) Then f is a constant on all of G as well, by the identity theorem. Thus we have proved what we wanted. Sorry about the mistake.

Problem 8: Let $g(z) = f(z)e^{-z}$. Then $|g(z)| \leq 1$ for all z on the unit circle. But the maximum value of $|g(z)|$ on the unit disk D is attained on the boundary, so it follows that $|g(z)| \leq 1$ for all $z \in D$. In fact, this inequality must be strict, since $|g(z)|$ cannot have a local maximum in D. We're almost ready to use Schwarz's lemma for g, but it doesn't send 0 to 0. The solution, as usual, is to use LFTs. Define

$$\phi(z) = \frac{z + \ln 2}{1 + z \ln 2}, \quad h = g \circ \phi^{-1}$$

Then h is a holomorphic function mapping D to itself, and $h(0) = 0$. Now we have that $|h(z)| \leq |z|$ for all $z \in D$. So now let $z = \phi(\ln 2)$. Then we get

$$|g(\ln 2)| \leq |\phi(\ln 2)| = \frac{2 \ln 2}{1 + (\ln 2)^2},$$

so

$$|f(\ln 2)| \leq |g(\ln 2) e^{\ln 2}| = \frac{4 \ln 2}{1 + (\ln 2)^2}$$

This is our answer. To find an f which attains this maximum absolute value, we need our inequalities to be equalities, so that $h(z) = \lambda z$ for some constant λ on the unit circle. To make life easy, we might as well take $\lambda = 1$. Then

$$f(z) = \phi(z) e^z,$$

where ϕ is the LFT we defined above, is a suitable choice for f.

I really like this problem; it combines the maximum modulus principle and all aspects of Schwarz's lemma with some basics about LFTs. If you can solve this, then you understand a lot of what we've been doing in the last few weeks.