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Solution Set 7

Problem 1: As is made clear in VII.9, we just need to show the identity

$$\frac{h^{(n)}(z_0)}{n!} = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} \cdot \frac{g^{(n-k)}(z_0)}{(n-k)!}$$

for all nonnegative integers n. The base case, $n = 0$, is immediate. The case $n = 1$ is just the product rule. Now suppose this formula is true for n. Then

$$h^{(n)}(z_0) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z_0) g^{(n-k)}(z_0)$$

and taking one more derivative of both sides (and letting $\ell = k + 1$), we get

$$\begin{aligned} h^{(n+1)}(z_0) &= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k+1)}(z_0) g^{(n-k)}(z_0) + f^{(k)}(z_0) g^{(n-k+1)}(z_0) \right) \\ &= \sum_{\ell=1}^{n+1} \binom{n}{\ell-1} f^{(\ell)}(z_0) g^{(n+1-\ell)}(z_0) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z_0) g^{(n+1-k)}(z_0) \end{aligned}$$

But now we can combine the two sums by changing the variables ℓ and k to m, and since $\binom{n}{-1} = \binom{n}{n+1} = 0$, we can extend them both to sums from 0 to $n + 1$. Then we get

$$h^{(n+1)}(z_0) = \sum_{m=0}^{n+1} \left(\binom{n}{m-1} + \binom{n}{m} \right) f^{(m)}(z_0) g^{(n+1-m)}(z_0) = \sum_{m=0}^{n+1} \binom{n+1}{m} f^{(m)}(z_0) g^{(n+1-m)}(z_0)$$

by Pascal's identity. So the statement is true for $n + 1$, hence true for all nonnegative integers by induction.

Problem 2: (a) Well, by Cauchy's integral formula, we have that this integral equals $\frac{2\pi i f^{(37)}(0)}{37!}$, where $f(z) = \sin z$. The 37th derivative of $\sin z$ is $\cos z$, which evaluates to 1 at 0, so we get $\frac{2\pi i}{37!}$.

(b) Let $f(z) = \frac{1}{8}(z - 2)^3$. Then this integral equals

$$\frac{2\pi i f''(1/2)}{2!} = -\frac{9\pi i}{8}.$$

Problem 3: Rewriting as instructed and multiplying top and bottom by $e^{i\theta}$, we get

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta} - r(e^{2i\theta} + 1) + r^2 e^{i\theta}} \, d\theta = \frac{-1}{2\pi i r} \int_\gamma \frac{dz}{z^2 - (r + 1/r)z + 1}$$

where γ is the unit circle oriented counterclockwise. Now then, the denominator factors as $(z - r)(z - 1/r)$, so the integral equals

$$\frac{-1}{2\pi i r} \cdot \frac{2\pi i}{r - 1/r} = \frac{1}{1 - r^2}.$$

Problem 4: The Taylor series of $\operatorname{Log}(1 - z)$ is just $-\sum_{n=1}^{\infty} \frac{z^n}{n}$, because its derivative is $\frac{-1}{1-z}$ and it equals 0 at 0. So we want to take the Cauchy product of this series with itself. The coefficient of z^{n+1} in this product is

$$\sum_{k=1}^{n} \frac{1}{k(n+1-k)} = \frac{1}{n+1} \sum_{k=1}^{n} \left(\frac{1}{k} + \frac{1}{n+1-k} \right) = \frac{1}{n+1}(H_n + H_n)$$

and so we are done.

Problem 5: First note that being holomorphic is a local condition, so it is enough to check that if G is an open rectangle with sides parallel to the coordinate axes, and f has the given properties in G, then f is differentiable at the center z_0 of G. (We'll actually show that f is holomorphic in G. Any point in an arbitrary G can be covered by some open rectangle of this form.)

So let's assume that G is a disk with center z_0, and define $g(z) = \int_{\gamma_z} f(\zeta)\,d\zeta$, where γ_z is a path that describes two sides of the rectangle with sides parallel to the coordinate axes, and opposite vertices at z_0 and z, oriented so that it starts at z_0 and ends at z. Note that it doesn't matter which of the two such paths I pick, because of the given property–the difference between the two integrals is the integral of f over a rectangle contained in G. Anyway, we'll show that $g(z)$ is holomorphic on G, with $g' = f$.

For two points $z, z_1 \in G$, consider $g(z) - g(z_1)$. I claim that it equals

$$\int_{\gamma_{z_1,z}} f(\zeta)\,d\zeta$$

where $\gamma_{z_1,z}$ is again a path from z_1 to z which describes two sides of the rectangle with sides parallel to the coordinate axes and opposite vertices at z_1 and z. I'll have to leave the proof of this claim to you; the point is that if we go from z_0 to z_1 along a path consisting of a bunch of segments parallel to the coordinate axes, then the integral doesn't depend on our choice of the path, because of the given property. So if we go first from z_0 to z_1, then from z_1 to z, along such paths, then we get the same thing we would have gotten if we had gone from z_0 to z along the sides of a rectangle. Okay?

Anyway, we get

$$\frac{g(z) - g(z_1)}{z - z_1} - f(z_1) = \frac{1}{z - z_1}\int_{\gamma_{z_1,z}} (f(\zeta) - f(z_1))\,d\zeta$$

and now, for any $\epsilon > 0$, there is $\delta > 0$ such that if $|\zeta - z_1| < \delta$, then $|f(\zeta) - f(z_1)| < \epsilon/\sqrt{2}$. Suppose now that $|z - z_1| < \delta$. Then $|\zeta - z_1| < \delta$ for all $\zeta \in \gamma_{z_1,z}$, so we can bound the integrand above by $\epsilon/\sqrt{2}$:

$$\left|\frac{g(z) - g(z_1)}{z - z_1} - f(z_1)\right| < \frac{1}{|z - z_1|}\frac{\epsilon}{\sqrt{2}}(|z - z_1|\sqrt{2}) = \epsilon$$

because the length of $\gamma_{z_1,z}$ is at most $|z - z_1|\sqrt{2}$ (why?), so it follows that

$$\lim_{z\to z_1}\frac{g(z) - g(z_1)}{z - z_1} = f(z_1)$$

so that g is holomorphic in G with $g' = f$.

Problem 6: The condition implies that there are positive real numbers R and M such that if $|z| > R$, $\left|\frac{f(z)}{z^n}\right| < M$. Suppose $r > R$, and let C_r be the circle centered at 0 of radius r. Suppose $m > n$. Then

$$|f^{(m)}(0)| = \left|\frac{1}{2\pi i}\int_{C_r}\frac{f(z)}{z^{m+1}}\,dz\right| < \frac{1}{2\pi}\frac{Mr^n}{r^{m+1}}(2\pi r) = Mr^{n-m}$$

Letting r tend to ∞, we must have that $f^{(m)}(0) = 0$, for all $m > n$. So the Taylor series of f centered at 0 is just a polynomial of degree at most n, but the Taylor series of f centered at 0 represents f on all of $\mathbb{C}$. So we're done. (You might have done the problem by showing that $f^{(n+1)}$ vanishes on all of $\mathbb{C}$, as well, but this is easier.)

Problem 7: Let C_r be the circle of radius r centered at z. Suppose C_r lies inside the open unit disk. Use the Cauchy integral formula

$$f'(z) = \frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{(w - z)^2}\,dz$$

and get

$$|f'(z)| < \frac{1}{2\pi}\frac{1}{r^2}2\pi r = \frac{1}{r}$$

This holds for all r such that C_r lies inside the open unit disk. Certainly if $r < 1 - |z|$, then C_r must lie inside the open unit disk, so we can let r tend to $1 - |z|$ to obtain $|f'(z)| \leq \frac{1}{1-|z|}$.

You can use Schwarz's lemma to get better bounds on $|f'(z)|$; see exercise VII.17.3.

Problem 8: (a) It is immediately clear that $Q_k(z_k) = Q'(z_k)$ and $Q_j(z_k) = 0$ if $j \neq k$. So the polynomial

$$P(z) = \sum_{k=1}^{n} \frac{a_k Q_k(z)}{Q'(z_k)}$$

certainly satisfies the conditions we need. So we must only show uniqueness. Well, suppose P_1 and P_2 are two such polynomials, then $P_1 - P_2$ will be a polynomial of degree $< n$ with n distinct roots, namely the z_k. Uniqueness follows from the following

Lemma: A polynomial of degree $< n$ with n distinct complex roots must be the zero polynomial.

Proof: Let F be such a polynomial. By the Fundamental Theorem of Algebra, F factors into linear factors, $F(z) = a\prod_{k=1}^{d}(z - z_k)$. (We might have $d = 0$, in which case $F(z) = a$.) Suppose that $a \neq 0$. Then if $z \neq z_k$, $F(z) \neq 0$, so there are at most $d = \deg(F)$ roots of F. Therefore $a = 0$ and $F(z)$ is the zero polynomial. $\square$

Of course, I didn't need the full power of the Fundamental Theorem of Algebra to prove this result. But this isn't Math 113, so we aren't that concerned with things like that.

(b) Here $Q(z) = (z-1)(z+1)(z-2)$ and we get

$$P(z) = \frac{3(z+1)(z-2)}{-2} + \frac{(z-1)(z-2)}{6} + \frac{7(z-1)(z+1)}{3} = z^2 + z + 1.$$

Success!