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Solution Set 6

Problem 1: Notice that $\text{Log}(w)$ is defined and holomorphic in the disc $|w-1| < 1$. Let $\alpha(z) = \text{Log}(f(z))$; then α is defined and holomorphic in G, by the Chain Rule. We have already seen that $\frac{d}{dw}\text{Log}(w) = 1/w$, so then $\alpha'(z) = f'(z)/f(z)$ by the Chain Rule, and thus we are done by the Fundamental Theorem of Calculus.

Problem 2: If we expand the binomial, we'll get a sum of various (positive and negative) powers of z. The only one that will integrate to something nonzero over γ is z^{-1}. The coefficient of z^{-1} in the expansion is pretty clearly $\binom{2n}{n}$. So we have

$$2\pi i\binom{2n}{n} = \int_\gamma \frac{1}{z}\left(z+\frac{1}{z}\right)^{2n} dz = \int_0^{2\pi} e^{-it}\left(e^{it}+e^{-it}\right)^{2n} ie^{it}\,dt = 2^{2n} i\int_0^{2\pi} \cos^{2n} t\,dt$$

and so it follows that

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n} t\,dt = \frac{1}{2^{2n}}\binom{2n}{n} = \frac{(2n)!}{(2\cdot 4\cdots(2n))^2} = \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}$$

Problem 3: For any $\epsilon > 0$, we can pick $\delta > 0$ such that if $|z-z_0| < \delta$, $|f(z)-f(z_0)| < \epsilon/2\pi$. Then if $r < \delta$,

$$\left|\int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| < \frac{\epsilon/2\pi}{r} 2\pi r = \epsilon,$$

so we see that

$$\lim_{r\to 0}\int_{C_r} \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0) = \lim_{r\to 0}\int_{C_r}\frac{f(z)-f(z_0)}{z-z_0}\,dz = 0.$$

Problem 4: One way to do this is to look at the proof of Cauchy's theorem (beta). Let D_r be the interior of C_r. If you let $f = u+iv$, then looking at your notes gives the first equality of the following series of equalities:

$$\begin{aligned}\frac{1}{r^2}\int_{C_r} f(z)\,dz &= \frac{1}{r^2}\iint_{D_r} [(-v_x-u_y)+i(u_x-v_y)]\,dA \\ &= \frac{1}{\pi r^2}\iint_{D_r} 2\pi i\frac{\partial f}{\partial \overline{z}}(x,y)\,dA\end{aligned}$$

which equals the average value of the integrand on D_r. As $r\to 0$, this clearly goes to $2\pi i\frac{\partial f}{\partial \overline{z}}(z_0)$, because $\frac{\partial f}{\partial \overline{z}}$ is continuous.

Another way to do this, which is a lot more elegant, is to show that if f is C^1, then

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)-\frac{\partial f}{\partial z}(z_0)(z-z_0)-\frac{\partial f}{\partial \overline{z}}(\overline{z}-\overline{z_0})}{z-z_0} = 0$$

but I do not know if there is an easy way to prove this rigorously without writing everything out in terms of u, v, x, y. If you look at it for a minute, you should be convinced that it is true, if you think it's reasonable to view complex functions as functions of z and $\overline{z}$ instead of x and y. (You can change variables back and forth pretty easily.) Anyway, once you establish this, consider what happens when you integrate this fraction divided by $\overline{z}-\overline{z_0}$ over C_r, and use the previous exercise. I'll leave the details to you.

For the second part, simply check that $\frac{\partial f}{\partial \overline{z}}(z_0) = 0$ if and only if f satisfies the Cauchy-Riemann equations at z_0; since f is C^1, this is equivalent to f being differentiable at z_0.

Problem 5: Let γ_1 be the edge of the triangle on the real axis. Let γ_2 be the edge of the triangle not touching the origin. Let γ_3 be the remaining edge. Orient them so that the triangle is oriented counterclockwise. Then immediately we see that $\lim_{R\to\infty} \int_{\gamma_1} e^{-z^2}\,dz = \sqrt{\pi}/2$.

A parameterization for γ_2 is $\gamma_2(t) = R(1+\alpha t)$, $t \in [0,1]$, where $\alpha = e^{\pi i/8} - 1$. So

$$\left|\int_{\gamma_2} e^{-z^2}\,dz\right| = \left|\int_0^1 e^{-R^2(1+\alpha t)^2} R\alpha\,dt\right| \le e^{-R^2 a} R|\alpha|,$$

where $a = \min\{\mathrm{Re}((1+\alpha t)^2)\colon t \in [0,1]\}$. The only thing that matters is that a is positive. (The argument of $1+\alpha t$ is between 0 and $\pi/8$, so its square has argument between 0 and $\pi/4$.)

At any rate, it is clear that this integral is bounded above in absolute value by something approaching 0 as $R \to \infty$, so it must approach 0 as $R \to \infty$.

Now a parameterization for $-\gamma_3$ is simply $(-\gamma_3)(t) = te^{\pi/8}$, $t \in [0, R]$, and by Cauchy's theorem and what we just did, we must have that $\lim_{R\to\infty} \int_{\gamma_1} e^{-z^2}\,dz = \lim_{R\to\infty} \int_{-\gamma_3} e^{-z^2}\,dz$. So we get

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-t^2 e^{\pi i/4}} e^{\pi i/8}\,dt$$

and, putting the constant on the other side and taking real parts, we get

$$\cos(\pi/8)\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-t^2/\sqrt{2}} \cos(t^2/\sqrt{2})\,dt$$

Letting $u = t/2^{1/4}$, we get

$$2^{-1/4}\cos(\pi/8)\sqrt{\pi}/2 = \int_0^\infty e^{-u^2}\cos(u^2)\,du$$

and I will let you discover why $\cos(\pi/8) = 2^{-3/4}\sqrt{1+\sqrt{2}}$ (hint: double-angle formula).

Problem 6: As I mentioned in class, it's silly not to integrate over a sector instead of a triangle, so this is what I'll do. Again, make $\gamma_1, \gamma_2, \gamma_3$ just as in the previous exercise (except now γ_2 is an arc and not a line). Again we have $\lim_{R\to\infty} \int_{\gamma_1} e^{-z^2}\,dz = \sqrt{\pi}/2$. Now

$$\int_{\gamma_2} e^{-z^2}\,dz = \int_0^{\pi/4} e^{-R^2\cos 2\theta - iR^2 \sin 2\theta} ie^{i\theta}\,d\theta,$$

so we get that

$$\left|\int_{\gamma_2} e^{-z^2}\,dz\right| \le \int_0^{\pi/4} e^{-R^2\cos 2\theta}\,d\theta \le \int_0^{\pi/4} e^{-R^2(1-4\theta/\pi)}\,d\theta = \frac{\pi}{4R^2}(1-e^{-R^2})$$

which goes to 0 as $R \to \infty$. Note that I used the inequality $\cos 2\theta \ge 1 - 4\theta/\pi$, $\theta \in [0, \pi/4]$. If you want to see why this is true, try drawing the graph of the left and right sides.

Just as in the last problem, then, Cauchy's theorem gives that

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-(e^{\pi i/4}t)^2} e^{\pi i/4}\,dt = e^{\pi i/4}\int_0^\infty e^{-it^2}\,dt.$$

So we can equate real and imaginary parts of the resulting equation

$$e^{-\pi i/4}\frac{\sqrt{\pi}}{2} = \int_0^\infty \cos(t^2)\,dt - i\int_0^\infty \sin(t^2)\,dt$$

to obtain

$$\int_0^\infty \cos(t^2)\,dt = \int_0^\infty \sin(t^2)\,dt = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

Problem 7: First of all, if $b > 1$ then $(1+z^2)^{-1}$ is not holomorphic inside the given rectangle. (We will still be able to evaluate this integral once we learn the residue theorem, which will come later.) Now we make $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ just as we did in the class example with a rectangle. The integral over γ_1 approaches

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

as $a \to \infty$. Using the identity

$$\left|\frac{1}{1+z^2}\right| \leq \frac{1}{|z|^2-1} \leq \frac{1}{a^2-1}$$

for $z \in \gamma_2$ or γ_4 (as long as $a > 1$), we quickly see that the integral of $(1+z^2)^{-1}$ on either of those curves is bounded in absolute value by $\frac{\sqrt{b}}{a^2-1}$, which goes to 0 as $a \to \infty$.

So Cauchy's theorem allows us to conclude that

$$\pi = \int_{-\infty}^{\infty} \frac{dt}{1+(t+i\sqrt{b})^2} = \int_{-\infty}^{\infty} \frac{1+(t-i\sqrt{b})^2}{|1+(t+i\sqrt{b})^2|^2}\, dt$$

and we can take real parts of both sides (if we expanded out, we'd get that the imaginary part was 0 anyway, because it would end up being the integral of an odd function). This yields

$$\pi = \int_{-\infty}^{\infty} \frac{1-b+t^2}{(1-b+t^2)^2+(2t\sqrt{b})^2}\, dt = \int_{-\infty}^{\infty} \frac{1-b+t^2}{(1-b+t^2)^2+4bt^2}\, dt$$

Problem 8: Oh good, an easy one! The function $g(\zeta) = \frac{f(\zeta)}{\zeta - z}$ is holomorphic in an open disk containing C and the region enclosed by C, because the denominator only vanishes at one point outside this region. (Just take a disk centered at the center of C, with radius bigger than C's radius, but smaller than the distance from the center of C to z.) Then by Cauchy's theorem for a convex region, we have $\int_C g(\zeta)\, d\zeta = 0$.