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Solution Set 5

Problem 1: If $|z| < \min(R_1, R_2)$, then $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ both converge, so then $\sum_{n=0}^{\infty}(a_n+b_n)z^n$ converges as well. So the radius of convergence of this series is at least $\min(R_1, R_2)$. (You can surely work out what this means when one or both of the radii is ∞.)

And for the second part, if R is the radius of convergence of $\sum_{n=0}^{\infty} a_n b_n z^n$, we have (by property (ii) on p. 52):

$$\frac{1}{R} = \limsup_{n\to\infty} a_n b_n \le \frac{1}{R_1}\frac{1}{R_2}$$

so $R \ge R_1 R_2$ (provided this product is well-defined).

Problem 2: It is easier to show that $\sum_{n=1}^{\infty} n a_n z^n$ has the same radius of convergence; it should be clear that dividing a series (with no constant term) by z does not affect its radius of convergence, so this is good enough. Suppose the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is R.

This follows directly from the following computation:

$$\limsup_{n\to\infty} |n a_n|^{1/n} = \lim_{n\to\infty} n^{1/n} \cdot \limsup_{n\to\infty} |a_n|^{1/n} = 1 \cdot \frac{1}{R} = \frac{1}{R}$$

by property (ii) on p. 52. (I'm omitting the proof that $n^{1/n} \to 1$ here, but you should probably have put it on your homework, to remind yourself and the reader that you know why it's true–good old L'Hopital.)

Problem 3: (a) 1, by the ratio test.

(b) If we replaced z^{3n} by z^n, the answer would be 27, by the ratio test:

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = \lim_{n\to\infty} \frac{(n!)^3(3n+3)!}{(3n)!((n+1)!)^3} = \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3} = 27.$$

Now, if $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R, then $\sum_{n=0}^{\infty} a_n z^{3n}$ has radius of convergence $R^{1/3}$ (compare exercise V.13.3), which you can check by noting that it converges for $|z^3| < R$ and diverges for $|z^3| > R$. So the final answer is 3.

(c) It is not hard to see that the radius is the reciprocal of

$$\limsup_{n\to\infty} \left(\frac{1}{n}\right)^{1/n!} = \limsup_{n\to\infty} \left(\left(\frac{1}{n}\right)^{1/n}\right)^{1/(n-1)!},$$

but the thing in parentheses goes to 1 as $n \to \infty$, and the exponent goes to 0, so this sequence has a limit of 1, which must equal its lim sup. Then the answer is $1/1 = 1$.

(d) It should be clear that the sequence $\left((n!)^{1/n!}\right)$ is just a subsequence of $\left(n^{1/n}\right)$, which converges to 1, so it must also converge to 1. So $R = 1$.

(e) Well, we want the reciprocal of

$$\limsup_{n\to\infty} (n^n)^{1/n^2} = \limsup_{n\to\infty} n^{1/n} = 1,$$

so $R = 1$.

Problem 4: This is a tricky one. Let $z = e^{i\theta}$. Assume that $\theta \in (0, 2\pi)$. Then the real part of the series is $\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{n}$, and we'd like to show that this converges if θ is not a multiple of 2π. The proof when we replace cos by sin will be similar.

Anyway, we will use a criterion for convergence due to Dirichlet: Suppose the partial sums of a series $\sum b_n$ are all bounded, and suppose that c_n is a decreasing sequence of positive numbers tending to 0. Then $\sum b_n c_n$ converges. (I leave the proof to you; it's probably in any good real analysis book.)

Apply Dirichlet's criterion with $b_n = \cos(n\theta)$ and $c_n = 1/n$. By exercise I.10.3 (which you can easily get from de Moivre's theorem), the partial sums of $\sum b_n$ are bounded in absolute value by $\frac{1}{|\sin(\theta/2)|}$, and the same would be true if we had taken $b_n = \sin(n\theta)$ instead. (Think about it.) Note that we are using that θ is not a multiple of 2π.

Problem 5: Let us consider the coefficients of $z_1^j z_2^k$ on both sides of the purported equality. If they are the same for all j, k, then we are done.

Well, the coefficient of $z_1^j z_2^k$ on the left is simply $\frac{1}{j!k!}$, because the only way to get this monomial is by multiplying $\frac{z_1^j}{j!}$ by $\frac{z_2^k}{k!}$. To get the coefficient of $z_1^j z_2^k$ on the right, we must look only at $\frac{(z_1+z_2)^{j+k}}{(j+k)!}$; expanding the binomial, we get that the coefficient of $z_1^j z_2^k$ is

$$\frac{\binom{j+k}{j}}{(j+k)!} = \frac{1}{j!k!},$$

as desired.

Problem 6: By exercise V.6.1, it is enough to show that the partial sums $s_k(z)$ of $\sum g_n(z)$ are uniformly Cauchy (the absolute convergence follows immediately from conditions (1) and (2)). Well,

$$|s_{m+p}(z) - s_m(z)| = \left| \sum_{m+1}^{m+p} g_n(z) \right| \leq \sum_{m+1}^{m+p} |g_n(z)| \leq \sum_{m+1}^{m+p} M_n,$$

and since $\sum M_n$ converges, its partial sums satisfy the Cauchy condition; for all $\epsilon > 0$ there exists a positive integer N such that that last sum is always $< \epsilon$ for $m > N$. This immediately shows that the sequence $(s_k(z))$ is uniformly Cauchy, so we are done.

Problem 7: For all $\epsilon > 0$ there exists a positive integer N such that $|g_n(z) - g(z)| < \epsilon/3$ for $n \geq N$ and for all $z \in G$. Now pick a point $z \in G$. Then there exists $\delta > 0$ such that $|w - z| < \delta \Rightarrow |g_N(w) - g_N(z)| < \epsilon/3$. So then if $|w - z| < \delta$,

$$|g(w) - g(z)| \leq |g(w) - g_N(w)| + |g_N(w) - g_N(z)| + |g_N(z) - g(z)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

and we are done.