Patrick Corn
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Solution Set 4

Problem 1: (a) Suppose $\phi(C)$ is the real axis and $\overline{\phi(z)} = \phi(w)$. Let ψ be an LFT with $\psi(C)$ equal to the real axis. Then $\psi\phi^{-1}$ sends the real axis to itself, and so it can be represented by coefficients $a, b, c, d \in \mathbb{R}$. So it is clear that $\overline{\psi\phi^{-1}(z)} = \psi\phi^{-1}(\overline{z})$ (why?) Then

$$\overline{\psi(z)} = \overline{\psi\phi^{-1}(\phi(z))} = \psi\phi^{-1}(\overline{\phi(z)}) = \psi\phi^{-1}(\phi(w)) = \psi(w).$$

(b) This is obvious: because z and w are symmetric with respect to C, there is some LFT ψ sending z and w to conjugate points and C to the real axis. Then $\psi\phi^{-1}$ sends $\phi(z)$ and $\phi(w)$ to conjugate points and $\phi(C)$ to the real axis, so we are done.

(c) When C is the unit circle, we know that the map $\alpha(z) = \frac{i(z-1)}{z+1}$ sends C to the real axis, and $\alpha(0) = -i$ and $\alpha(\infty) = i$. So these points are symmetric with respect to the unit circle. Now if C is the circle $|z-a| = R$, the map $z \mapsto Rz + a$ sends the unit circle to C and sends $0 \mapsto a$, $\infty \mapsto \infty$. So a and ∞ are symmetric with respect to C, by the symmetry principle.

(d) Let us figure out $\phi(\infty)$ by figuring out which point $2i$ is symmetric to. Plugging in $z-1$ for z in the map α of part (c) gives an LFT $\psi(z) = \frac{i(z-2)}{z}$ sending $|z-1| = 1$ to the real axis. Now $\psi(2i) = -1 + i$, so we need to solve for w such that $\psi(w) = -1 - i$. A bit of computation gives $w = \frac{4+2i}{5}$. So this is $\phi(\infty)$, by the symmetry principle. Since ϕ is now determined by the images of these three points, we can figure it out by the following computation (which is certainly not the only way to derive it, just my chosen way):

Because $\phi(-1) = 0$ and $\phi(0) = 2i$, we know that ϕ is of the form

$$\phi(z) = \frac{2i(z+1)}{cz+1}$$

for some c. Then looking at $\phi(\infty)$ yields $\frac{2i}{c} = \frac{4+2i}{5}$, so $c = 1 + 2i$. We get

$$\phi(z) = \frac{2i(z+1)}{(1+2i)z+1}$$

(e) An obvious LFT sending $|z-a| = R$ to the real axis is just the composition of $z \mapsto \frac{z-a}{R}$ with α, so we get

$$\psi(z) = \frac{i\left(\frac{z-a}{R} - 1\right)}{\frac{z-a}{R} + 1} = \frac{i(z-a-R)}{z-a+R}$$

Now by part (a), we must have $\psi(z) = \overline{\psi(w)}$, so we get

$$\frac{i(z-a-R)}{z-a+R} = \frac{-i(\overline{w} - \overline{a} - R)}{\overline{w} - \overline{a} + R}$$

$$(z-a)(\overline{w} - \overline{a}) - R(\overline{w} - \overline{a}) + R(z-a) - R^2 = -(z-a)(\overline{w} - \overline{a}) - R(\overline{w} - \overline{a}) + R(z-a) + R^2$$

$$2(z-a)(\overline{w} - \overline{a}) = 2R^2$$

and now cancelling the 2 gives us the equation of the hint. What does this equation mean? Taking absolute values gives immediately that the geometric mean of $|z-a|$ and $|w-a|$ is R, and since the product of $z-a$ and the conjugate of $w-a$ is positive and real, they must be positive multiples of each other by exercise I.4.2. Thus they are parallel vectors, so the three points are collinear (and indeed z and w are not on opposite sides of a).

Problem 2: The image of the real axis under the map $z \mapsto \frac{z+1}{z-1}$ is the real axis. The image of $[-1, 1]$ is the segment starting at 0 and ending at ∞, going through -1 (which is the image of 0); in other words, it is the negative real axis. Composing the principal branch of $\sqrt{z}$ with the LFT

$z \mapsto \frac{z+1}{z-1}$ gives a branch of $\sqrt{\frac{z+1}{z-1}}$ defined on G. Multiplying this branch by $z-1$ gives a branch of $\sqrt{z^2-1}$ in G.

Problem 3: We have

$$\begin{aligned}(1+i)^{1+i} = e^{(1+i)\log(1+i)} &= e^{(1+i)(\ln\sqrt{2}+\pi i/4+2\pi in)} \\ &= e^{\ln\sqrt{2}}e^{-\pi/4-2\pi n}e^{i(\ln\sqrt{2}+\pi/4+2\pi n)} \\ &= (1+i)(\cos(\ln\sqrt{2})+i\sin(\ln\sqrt{2}))e^{-\pi/4-2\pi n},\ n\in\mathbb{Z}.\end{aligned}$$

Problem 4: Suppose there is a branch $s(z)$ of $z^{1/n}$ on the region. Let N be the region minus the negative real axis. Then there is a principal branch $t(z)$ of $z^{1/n}$ on N, defined by $t(re^{i\mathrm{Arg}z}) = r^{1/n}e^{i(\mathrm{Arg}z)/n}$. Of course, t is never zero on N, and the function $(s|_N)(z)/t(z)$ has image inside the discrete set of nth roots of 1 in $\mathbb{C}$. It's continuous and N is connected, so it is a constant function. Thus $s|_N(z) = \zeta t(z)$ for some ζ with $\zeta^n = 1$.

Now consider the sequences of points $z_n = -1/4+i/n$ and $w_n = -1/4-i/n$. Clearly $t(z_n) \to i/2$ and $t(w_n) \to -i/2$, so $s(z_n)$ and $s(w_n)$, which should both approach $s(1/4)$, go to $\zeta i/2$ and $-\zeta i/2$, respectively. This is absurd (of course $\zeta \neq 0$), so there is no such s.

Problem 5: First suppose that (g_n) converges uniformly. Then for all $\epsilon > 0$, there exists a positive integer N such that $|g(z) - g_n(z)| < \epsilon/2$ whenever $n \geq N$, for all $z \in S$. Then for $m, n \geq N$, we have

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g(z)| + |g(z) - g_m(z)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $z \in S$, so that the sequence is uniformly Cauchy.

Next suppose that (g_n) is uniformly Cauchy. Since $(g_n(z))$ is a Cauchy sequence for any $z \in S$, it must converge to some value $g(z)$. So (g_n) converges pointwise to g; we will show that the convergence is uniform.

Pick an $\epsilon > 0$, and find N such that $|g_n(z) - g_m(z)| < \epsilon/3$ whenever $m, n \geq N$. Note that the values $\{g_n(z) \colon n \geq N\}$ are contained in a closed disk of radius $\epsilon/3$ around $g_N(z)$. Therefore their limit $g(z)$ is contained in that disk as well. So $|g_n(z) - g(z)| \leq$ the diameter of that disk, which is $2\epsilon/3 < \epsilon$. This is what we wanted.

Problem 6: Suppose (g_n) converges locally uniformly in G to g. Then each point $z \in G$ has a neighborhood U_z on which (g_n) converges uniformly. Let K be a compact subset of G. Then the union of the sets U_z, as z runs over all the points in K, must of course equal K. Because K is compact, there are finitely many neighborhoods U_{z_i}, $1 \leq i \leq k$, which cover K. Given $\epsilon > 0$, there is a positive integer N_i such that $|g(z) - g_n(z)| < \epsilon$ for $z \in U_{z_i}$ and $n \geq N_i$, so if we let $N = \max(N_i)$, then $|g(z) - g_n(z)| < \epsilon$ for all $z \in K$ and $n \geq N$.

Now suppose (g_n) converges uniformly on compact subsets. Because G is open, there is a neighborhood $|z - z_0| < \epsilon$ contained in G, for any point $z_0 \in G$. Then the closed disk $|z - z_0| \leq \epsilon/2$ is contained completely in G, and (g_n) must converge uniformly on this compact set. So it converges uniformly on the subset $|z - z_0| < \epsilon/2$. Therefore every point in G has a neighborhood on which the sequence converges uniformly.