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Solution Set 3

Problem 1: (a) The LFT $\phi(z) = \frac{z+1}{z-1}$ sends the extended real axis to the extended real axis, and squaring sends the extended real axis to the ray $[0, \infty)$.

(b) The LFT ϕ sends the extended imaginary axis to $|z| = 1$ (check this), and squaring sends $|z| = 1$ to itself (it wraps the unit circle around itself twice, if that makes any sense).

(c) By standard topology arguments, ϕ must send the right half plane to either the interior or exterior of the unit disk. Because $\phi(1) = \infty$, it must go to the exterior. Squaring sends the exterior of the unit disk to itself. (Why?) So the answer is $|z| > 1 \cup \{\infty\}$.

Problem 2: Note that it is not enough to show that Ae^{cz} satisfies this equation. You must show that this is the most general function that satisfies the equation. Basically, if you didn't do the problem the way I'm doing it, you did it wrong.

Suppose $f(z)$ satisfies the equation, and consider the function $g(z) = f(z)e^{-cz}$. Then $g'(z) = e^{-cz}(f'(z) - cf(z)) = 0$, and we have seen on a previous problem set that this implies that $g(z)$ is constant, say A. Then $f(z) = Ae^{cz}$ is the most general solution of the equation (since it is obvious that any f of this form satisfies the equation).

Problem 3: The same comments as in the previous solution apply here. Suppose $f'' = f$. Then $(f + f')' = f' + f'' = f + f'$, so by the previous problem we get $f(z) + f'(z) = Ae^z$. Similarly $(f - f')' = f' - f'' = f' - f$, so by the previous problem we get $f(z) - f'(z) = Be^{-z}$. Solving, we get $f(z) = \frac{A}{2}e^z + \frac{B}{2}e^{-z} = a \cosh z + b \sinh z$ where $a = \frac{A+B}{2}$, $b = \frac{A-B}{2}$. And it's clear that any function f of this form satisfies the equation, so we are done.

Problem 4: Let $y = e^{iz}$. Then $\cos z = \frac{y+1/y}{2}$, so we get the equation

$$y^2 + 1 = 4y \Rightarrow y = 2 \pm \sqrt{3}.$$

Let $z = x + iy$, $x, y \in \mathbb{R}$. Then we have $e^{-y}e^{ix} = 2 \pm \sqrt{3}$, so that x is an integer multiple of 2π and $y = \pm \ln(2 + \sqrt{3})$ (since the two values $2 \pm \sqrt{3}$ are reciprocals). Putting it together, we get

$$z = 2\pi n \pm i \ln(2 + \sqrt{3}), \ n \in \mathbb{Z}$$

Problem 5: Not really. The set of values of $\log(i^2)$ is $i(\pi + 2\pi n)$, $n \in \mathbb{Z}$. The set of values of $2\log(i)$ is $2i(\pi/2 + 2\pi n) = i(\pi + 4\pi n)$, $n \in \mathbb{Z}$. So the set of values of $2\log(i)$ is a subset of the set of values of $\log(i^2)$, but the two sets are not the same.

Problem 6: (a) Any good real analysis book will have the proof you need. It's good to know how to do these types of problems, so I will give a proof here.

First suppose that f is continuous. Suppose $\lim_{n\to\infty} z_n = L$. The first fact implies that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|z - L| < \delta$, $|f(z) - f(L)| < \epsilon$. The second fact implies that for all $\delta > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $|z_n - L| < \delta$. Putting the two facts together, start with ϵ, pick the δ, then pick the N, and conclude that if $n > N$, $|f(z_n) - f(L)| < \epsilon$.

Next suppose that f is not continuous at a point $a \in G$. So there is some $\epsilon > 0$ such that there is no δ satisfying $|z - a| < \delta \Rightarrow |f(z) - f(a)| < \epsilon$. Fix this ϵ. Now, taking $\delta = 1/n$, there is some number $z_n \in G$ with $|z_n - a| < 1/n$ and $|f(z_n) - L| \geq \epsilon$. Now it should be clear that the sequence (z_n) converges to a (why?) and just as clear that $f(z_n)$ does not approach $f(a)$. So f doesn't have the sequence property.

(b) Take a point $z_0 \in \mathbb{C}$ and consider the sequence $z_n = z_0/2^n$. Note that $f(z_n) = f(z_0)$, and since $z_n \to 0$, we must have $f(z_n) \to f(0)$. But $(f(z_n))$ is a constant sequence, so its limit equals that constant, which is $f(z_0)$. So $f(z_0) = f(0)$. This holds for all z_0, so we are done.