Patrick Corn
Math 185, 6/28/05

Solution Set 2

Problem 1: Let a and b be two points in U. Let γ be a differentiable path from a to b. Consider $h = f \circ \gamma \colon [0,1] \to \mathbb{C}$. Let $\gamma = \alpha + i\beta$, and let $f = u + iv$, and let $h = h_1 + ih_2$. Then h_1 and h_2 are differentiable (in the Math 1A sense!) with derivatives given by

$$h_1'(t_0) = u_x(x_0, y_0)\alpha'(t_0) + u_y(x_0, y_0)\beta'(t_0)$$
$$h_2'(t_0) = v_x(x_0, y_0)\alpha'(t_0) + v_y(x_0, y_0)\beta'(t_0)$$

where $x_0 + iy_0 = \gamma(t_0)$. This is just the chain rule.

The point is, however, that the partial derivatives of u and v are all zero because f' is identically zero. This implies that h_1' and h_2' are identically zero. Now this, in turn, implies that h_1 and h_2 are constant (you may even remember the proof: good old Mean Value Theorem). So h is constant. Then $h(0) = h(1)$, so $f(a) = f(b)$. This holds for any two points in U, so we're done.

As for the extra credit, ask me if you're interested. The hard part is the word "differentiable."

Problem 2: On the real and imaginary axes, $f(z)$ is identically zero. A moment's thought, then, will convince you that $u_x = u_y = v_x = v_y = 0$ at $(0,0)$. But

$$\lim_{x \to 0} \frac{f(x + ix)}{x + ix} = \lim_{x \to 0} \frac{|x|}{x(1 + i)}$$

(where now $x \in \mathbb{R}$) doesn't even exist. (It's approaching different values from the left and right). So the derivative of f at 0 cannot exist.

Problem 3: Think of $x = r\cos\theta$ and $y = r\sin\theta$ as intermediate variables. So then u and v are functions of x and y, which are in turn functions of r and θ. This is a perfect situation for the chain rule. So:

$$u_r = u_x x_r + u_y y_r = u_x \cos\theta + u_y \sin\theta$$
$$v_\theta = v_x x_\theta + v_y y_\theta = -r v_x \sin\theta + r v_y \cos\theta$$
$$u_\theta = u_x x_\theta + u_y y_\theta = -r u_x \sin\theta + r u_y \cos\theta$$
$$v_r = v_x x_r + v_y y_r = v_x \cos\theta + v_y \sin\theta$$

and then it is easy to see that $u_x = v_y$ and $u_y = -v_x$ imply $ru_r = v_\theta$ and $u_\theta = -rv_r$. (There is a minor subtlety: the latter two equations also imply the standard Cauchy-Riemann equations, and to completely rigorously answer the question, you should also show this. But I'm not too worried about this.)

Problem 4: (a) This should follow directly from problem 1, because the line segment between any two points in D is certainly a differentiable path (at least when we give it the right parameterization).

(b) The condition implies that $v = 0$, so that $v_x = v_y = 0$. Then the Cauchy-Riemann equations give $u_x = u_y = 0$ also. So f' is identically zero and we've reduced to part (a).

(c) I was wrong about polar Cauchy-Riemann–it doesn't apply here at all. Let's do it right. The condition is equivalent to saying that $u^2 + v^2$ is constant in D. We can assume that this constant is nonzero, because otherwise there is nothing to prove–if $|f| = 0$ everywhere, then f is identically zero. Taking partials, we get

$$\begin{aligned} 2uu_x + 2vv_x &= 0 \\ 2uu_y + 2vv_y &= 0 \end{aligned}$$

and then we can use the Cauchy-Riemann equations. We get

$$\begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The determinant of the above matrix is $4(u^2 + v^2)$, which is a (nonzero) constant! Then we can multiply by the inverse of the first matrix to get $u_x = u_y = 0$. This implies that $v_x = v_y = 0$ by Cauchy-Riemann, and we are done, just as in part (b).

(d) Here we have that v/u is constant. (If u or v vanishes somewhere, then they vanish everywhere, by the requirement, and then we can proceed as in part (b). So we can assume that v and u are everywhere nonzero.) Taking partials, and multiplying out the denominators:

$$\begin{aligned} uv_x - vu_x &= 0 \\ uv_y - vu_y &= 0 \end{aligned}$$

After using Cauchy-Riemann, we get

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The determinant never vanishes (since we're already assuming u and v never vanish), so we can conclude that $v_x = v_y = 0$ and proceed as in part (b).

Problem 5: You should not just be showing that the Cauchy-Riemann equations hold, because of the asymmetry discussed in the text–we do not yet know that holomorphic functions have continuous first partial derivatives (though it is true). So the Cauchy-Riemann equations alone are not enough. Instead, we just use the limit definitions. We want to show that the limit of the difference quotient for g at $\overline{z_0}$ exists, for $z_0 \in G$. So:

$$\begin{aligned} \lim_{z \to \overline{z_0}} \frac{g(z) - g(\overline{z_0})}{z - \overline{z_0}} &= \lim_{z \to \overline{z_0}} \frac{\overline{f(\overline{z}) - f(z_0)}}{z - \overline{z_0}} \\ &= \lim_{w \to z_0} \overline{\left(\frac{f(w) - f(z_0)}{w - z_0} \right)} = \overline{f'(z_0)}. \end{aligned}$$

(Here $w = \overline{z}$.) So the limit exists and equals $\overline{f'(z_0)}$.

Problem 6: Well, $(u^2)_{xx} = (2uu_x)_x = 2(uu_{xx} + (u_x)^2)$. Similarly for u_{yy}. So, using that u^2 and u are harmonic, we get

$$0 = u_{xx} + u_{yy} = 2u(u_{xx} + u_{yy}) + 2((u_x)^2 + (u_y)^2) = 2((u_x)^2 + (u_y)^2),$$

and so $u_x = u_y = 0$ everywhere, so that u is a constant.

Problem 7: Let us first show that if ϕ preserves the upper half plane, then it is induced by a matrix with real entries and unit determinant. Well, in this case, the real axis maps to itself. So let $z_1, z_2, z_3 \in \overline{\mathbb{C}}$ be the three points on the extended real axis going to $0, 1, \infty$, respectively. From Step 2 of our proof, ϕ is induced by a matrix with real entries. (Just look at the formulas–they're also in the book on p. 27.)

Also, we have that $\phi(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$, so

$$\operatorname{Im} \phi(i) = \operatorname{Im} \frac{ai+b}{ci+d} = \operatorname{Im} \frac{(ai+b)(-ci+d)}{c^2+d^2} = \frac{ad-bc}{c^2+d^2}$$

which is positive if and only if the determinant $ad - bc$ is positive. So $ad - bc$ is positive. Lastly, we can divide all the entries by $\sqrt{ad-bc}$ to obtain a matrix with unit determinant which also determines the same LFT ϕ, so we are done.

As for the converse, suppose ϕ is induced by a matrix with real entries and unit determinant. Well, then clearly ϕ preserves the real axis. Now for a bit of topology: let H_+ and H_- be the upper and lower half planes, respectively. We know that $\phi(H_+)$ and $\phi(H_-)$ are contained either in H_+ and H_-. Injectivity of ϕ implies that the two half planes go to two different half planes, and surjectivity implies that the containments are equalities (just as in the example we did in class). So we know that $\phi(H_+) = H_+$ or H_-; the only question is, which one? Well, $\phi(i)$ has positive imaginary part, by the work we did above, so it must be H_+.

Problem 8: It is easy to show that such matrices induce LFTs sending the unit disk onto itself: if $|z| = 1$ then we can multiply the bottom by $\overline{z}$ without changing the absolute value, i.e.

$$\left| \frac{az+b}{\overline{b}z+\overline{a}} \right| = \left| \frac{az+b}{\overline{b}+\overline{az}} \right| = 1,$$

and also $0 \mapsto b/\overline{a}$, which has absolute value less than 1. Then the standard topological arguments we did in class and in Problem 7 show that the unit disk maps onto itself.

Now suppose ϕ maps the unit disk onto itself. Let $\phi(z) = \frac{az+b}{cz+d}$. By looking at $\phi(1), \phi(-1), \phi(i)$, we immediately get that $|a+b| = |c+d|$, $|-a+b| = |-c+d|$, and $|ai+b| = |ci+d|$. The first two equalities and the parallelogram equality (I.6) then imply that $|a|^2 + |b|^2 = |c|^2 + |d|^2$. Then if we square both sides of the first equality, expand, and cancel, we get $\operatorname{Re}(a\overline{b}) = \operatorname{Re}(c\overline{d})$.

Now if we square both sides of the third equality and expand out, we get

$$|a|^2 + |b|^2 + 2\operatorname{Re}(ai\overline{b}) = |c|^2 + |d|^2 + 2\operatorname{Re}(ci\overline{d})$$

so, cancelling and using the fact that $\operatorname{Re}(iw) = -\operatorname{Im}(w)$, we get that $\operatorname{Im}(a\overline{b}) = \operatorname{Im}(c\overline{d})$. Hence $a\overline{b} = c\overline{d}$.

So $|a||b| = |c||d|$. From this and the parallelogram law, we get $|a| + |b| = |c| + |d|$ and $|a| - |b| = \pm(|c| - |d|)$. Now if it's $+$, we get $|a| = |c|$ and $|b| = |d|$, but this is no good, because $0 \mapsto b/d$, so $|b|$ should be less than $|d|$. Therefore, we get $|a| = |d|$ and $|b| = |c|$. So let $\lambda = b/\overline{c}$; then $|\lambda| = 1$. Let α be a square root of λ. Now we have $d = \overline{a}\lambda, c = \overline{b}\lambda$. So then $d/\alpha = \overline{a/\alpha}$, $c/\alpha = \overline{b/\alpha}$. So dividing everything by α at least gives us a matrix that induces ϕ of the form

$$\begin{pmatrix} x & y \\ \overline{y} & \overline{x} \end{pmatrix},$$

and notice that the determinant, $|x|^2 - |y|^2$, must be positive, because the image of 0 should have absolute value less than 1, so $|x|$ should be bigger than $|y|$.

Finally, divide everything out by the square root of $|x|^2 - |y|^2$ to get a matrix of the right form. Whew. There are probably easier ways, but I'm not sure how they'd work.

Problem 9: We compute $\phi(0) = -1$, $\phi(-1) = \infty$, $\phi(1) = 0$, $\phi(i) = i$. So then $|z| = 1$ goes to the imaginary axis and the real axis goes to the real axis. Now (eschewing the standard topology stuff) the interior of the disk goes to the left half plane (by looking at $\phi(0)$) and the upper half plane goes to the upper half plane (by looking at $\phi(i)$). So the image of the intersection is exactly the second quadrant.

Problem 10: Let $|f(a)| = \epsilon > 0$. Then there exists a real number δ such that if $|z - a| < \delta$, $f(z) - f(a)| < \epsilon$. Let $N = \{z \in \mathbb{C} : |z - a| < \delta\}$. Then suppose $f(z) = 0$ for $z \in N$. By the triangle inequality, we would have

$$\epsilon = |f(a)| = |f(a) - f(z) + f(z)| \leq |f(a) - f(z)| + |f(z)| < \epsilon + 0,$$

which is absurd. So $f(z)$ is nonzero for all $z \in N$, as required.