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Solution Set 12

Problem 1: Suppose that $f(z) = (z - z_0)^m g(z)$ where $g(z)$ is holomorphic and nonzero in a neighborhood of z_0. Then, as we have seen a few times already, the residue of f'/f at z_0 is m. (Just use the product rule.) So if f has a zero of order k, the residue at it is k, and if f has a pole of order ℓ, the residue at it is $-\ell$. Now just apply the residue theorem.

Problem 2: Let z_0 be a zero of f. Let C_r be the circle of radius r around z_0, where r is small enough so that f has no zeroes other than z_0 on or inside C_r. Let $\epsilon = \inf\{|f(z)| \colon z \in C_r\}$. Then $\epsilon > 0$, and we can find $N \in \mathbb{N}$ such that $|f(z) - f_n(z)| < \epsilon \leq |f(z)|$ for all $n > N$ and $z \in C_r$. So then f and f_n have the same number of zeroes inside C_r (with multiplicities), by Rouché's theorem. Doing this process at each of the m zeroes, we see that f_n has at least m zeroes if f has at least m zeroes, for n greater than the maximum of the numbers N we obtain via the above process.

As for the corollary, fix $w \in G$, and consider the functions $f_n(z) - f(w)$, which converge to $f(z) - f(w)$. If the latter function has at least two zeroes, then $f_n(z) - f(w)$ has at least two zeroes for sufficiently large n, but this contradicts the hypothesis that f_n is univalent. So $f(z) - f(w)$ has one zero for all $w \in G$, so f is univalent.

Problem 3: Let L_1 be the line segment $\{it \colon |t| \leq R\}$. Let L_2 be the arc $\{Re^{it} \colon \pi/2 \leq t \leq 3\pi/2\}$. Let K be the closed half-disk bounded by L_1 and L_2. Let $f(z) = z + a$. Let $g(z) = z + a - e^z$. Suppose R is large enough so that the closed disk $|z + a| \leq 1$ is contained in the interior of K. Then on L_1 and L_2, we have

$$|f(z) - g(z)| = |e^z| \leq 1 < |z + a| = |f(z)|,$$

and $f(z)$ has exactly one zero in K, so $g(z)$ has exactly one zero there as well. As we let R get bigger, K increases to cover the entire left half plane, so the result follows. (If there was another zero, it would lie inside K for large enough R, which would be a contradiction of the above Rouché's Theorem argument.)

To see that this zero is on the real axis, note that $g(-a) = -e^{-a} < 0$ and $g(0) = a - 1 > 0$, so g has a zero in $(-a, 0)$ by the Intermediate Value Theorem.

Problem 4: (i) The LFT $\phi_1(z) = \frac{-i(z-1)}{z+1}$ should work.

(ii) The function $\phi_2(z) = -\phi(z)^2$ should work, since squaring sends the upper half plane to $\mathbb{C}$ minus the nonnegative real axis.

(iii) Try $\phi_3(z) = \frac{\operatorname{Log}(\phi_2(z)) + \pi i}{2\pi}$.

(iv) Try $\phi_4(z) = \sqrt{\phi_1(z)} = e^{\operatorname{Log}(\phi_1(z))/2}$. You might also use $e^{\pi\phi_3(z)/2}$, if you liked doing things sequentially.

(v) Try $\phi_5(z) = \frac{1-\phi_4(z)}{1+\phi_4(z)}$.

(vi) Try $\phi_6(z) = \phi_5(z)^2$. If you want an explicit formula for ϕ_6, you can get it if you want; knock yourself out.

Problem 5: Suppose $f(0) = 0$, $f'(0)$ is real and positive, and f maps the open unit disk onto itself. Well, then $g = f^{-1}$ has the same properties (because g' and f' are reciprocals). In fact, by Schwarz's lemma, both $f'(0)$ and $g'(0)$ are ≤ 1, but they are reciprocals, so they must both equal 1. Since equality holds, $f(z) = \lambda z$, but $\lambda = f'(0) = 1$, so we are done.

Problem 6: Let f be a conformal automorphism of $\mathbb{C}$, and consider the singularity of f at ∞. We have seen that if it is removable, then f is a constant, which is no good. We have also seen that if it is a pole, then f is a nonconstant polynomial. But the univalence of f implies that it factors as $a(z - b)^n$ for some n (because it can't have more than one zero, but it has to factor by the Fundamental Theorem of Algebra), and it is clear that this function is only univalent in a neighborhood of b if $n = 1$ (you can see it directly, or use the Local Mapping Theorem).

Since it is clear that nonconstant linear functions are indeed conformal automorphisms of $\mathbb{C}$, we must only show that the singularity at ∞ cannot be essential.

Suppose it is; fix $w_0 \in \mathbb{C}$, and for some $z_0 \in \mathbb{C}$ we know that $f(z_0) = w_0$. If ϵ is a sufficiently small positive number, then in some open neighborhood N of z_0, all the values w in $V = \{|w - w_0| < \epsilon\}$ are assumed by f exactly once. (This is the local mapping theorem.) But now let $U = \{|z| > R\}$, and take R sufficiently large so that $U \cap N = \emptyset$. Then $f(U)$ must intersect V by the Casorati-Weierstrass criterion, which contradicts the univalence of f. So we are done.

Problem 7: This is obvious from problem 5; if we look at $h(z) = f^{-1}(g(z))$, then h is a conformal automorphism of the unit disk with $h(0) = 0$ and $h'(0) = g'(0)/f'(0)$, which is real and positive. So then we can conclude that h is the identity, by problem 5.

Problem 8: Let g be the Riemann map from G to the unit disk. Suppose $g(z_0) = w_0$. Let $\phi(z) = \frac{z - w_0}{1 - \overline{w_0} z}$, and let $h(z) = \phi(g(z))$. Then $h(z_0) = 0$, and h is still a Riemann map from G to the unit disk (why?). Let $k = h^{-1}$. Then $k(0) = z_0$. Now we just let $\ell(z) = k(\lambda z)$ for some λ on the unit circle to adjust the argument of the derivative at 0. (So if $\arg k'(0) = \theta$, simply let $\lambda = e^{i(\theta_0 - \theta)}$.)

Problem 9: First note that

$$\sin^3 x = \frac{(e^{ix} - e^{-ix})^3}{-8i} = -\frac{1}{8i}(e^{3ix} - e^{-3ix}) + \frac{3}{8i}(e^{ix} - e^{-ix}) = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

and then consider $f(z) = \frac{\frac{3}{4}e^{iz} - \frac{1}{4}e^{3iz}}{z^3}$. Integrating around the contour on p. 122, we get from Cauchy's theorem that the whole integral is 0, and the sum of the two integrals on the real axis is

$$\int_\epsilon^R \frac{\frac{3}{4}e^{ix} - \frac{3}{4}e^{-ix} - \frac{1}{4}e^{3ix} + \frac{1}{4}e^{-3ix}}{x^3}\, dx$$

(watch those negative signs!) So the limit as $\epsilon \to 0^+$ and $R \to \infty$ of this sum is

$$\int_0^\infty 2i \frac{\sin^3 x}{x^3}\, dx.$$

Standard arguments show that the integral around C_R tends to 0. I'll leave them to you; they're exactly the same as in the book. (You'll have to demonstrate on the exam that you can do something like this, though!) What about the integral around C_ϵ? Well, the integral of a function which is holomorphic near 0 around C_ϵ must tend to 0, so we just need to figure out the principal part of the Laurent series of our function and integrate that. A computation shows that

$$\frac{\frac{3}{4}e^{iz} - \frac{1}{4}e^{3iz}}{z^3} = \frac{1}{2z^3} + \frac{3}{4z} + g(z),$$

where $g(z)$ is holomorphic near 0. Now, the integral of $\frac{1}{2z^3}$ around C_ϵ equals 0 by the Fundamental Theorem of Calculus (the antiderivative is even, and the endpoints of the arc are negatives of each other!) And the integral of $\frac{3}{4z}$ around C_ϵ is just $-\frac{3\pi i}{4}$, either by the Fundamental Theorem of Calculus or direct parameterization. (Check this!)

Putting it all together yields

$$2i \int_0^\infty \frac{\sin^3 x}{x^3}\, dx - \frac{3\pi i}{4} = 0,$$

and so we are done.