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Math 185, 8/4/05

Solution Set 11

Problem 1: Suppose $\overline{\mathbb{C}} \setminus G$ is connected. Let U_i be the connected components of G. I want to show that $\overline{\mathbb{C}} \setminus U_i$ is connected, and for that it suffices to show that any point $z_0 \in U_j$ can be connected by a polygonal path to any point in $\overline{\mathbb{C}} \setminus G$. For that, it suffices to show that any point $z_0 \in U_j$ can be connected by a polygonal path to a point on ∂U_j (note: ∂U_j is not empty, unless U_j is empty or all of $\mathbb{C}$, both of which are rather trivial cases!) But this last part is clear from the connectedness of U_j (points on the boundary can be connected to some points inside U_j, via a neighborhood). So I'm done.

Suppose every connected component is simply connected. Then any contour Γ_i in U_i has winding number 0 around every point in $\overline{\mathbb{C}} \setminus U_i$. Since contours Γ in G are just sums of closed curves, and each closed curve is contained in some U_i, it follows that any contour Γ in G has winding number 0 around every point in the intersection of all the sets $\overline{\mathbb{C}} \setminus U_i$, which is $\overline{\mathbb{C}} \setminus G$. Then we can follow the second half of the proof of the winding number criterion to see that $\overline{\mathbb{C}} \setminus G$ must be connected.

Problem 2: Call this set S. Let U be a connected component of S, and suppose we have a contour Γ in U. Let $K = \mathbb{C} \setminus \text{ext}(\Gamma) = \text{int}(\Gamma) \cup \Gamma$. Then K is compact. Note that $K \subseteq G$, because G is simply connected. But now $|f|$ attains its maximum on K on $\partial K \subseteq \Gamma \subseteq U$, so $|f(z)| < c$ for all $z \in K$. Therefore $K \subseteq S$. But since every connected component of K intersects U nontrivially (on the boundary), every connected component of K is contained in U. So $K \subseteq U$. Therefore U contains the interior of every contour Γ in U, which implies that U is simply connected by the winding number criterion.

Problem 3: For $z_0 \notin G$, let $f(z) = z - z_0$. Then by assumption there is a branch of $\log(z - z_0)$ in G, and its derivative is $\frac{1}{z-z_0}$. So then

$$\text{ind}_\Gamma(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z - z_0}\, dz = 0$$

for all contours Γ in G, by the Fundamental Theorem of Calculus, so then G is simply connected by the winding number criterion.

Problem 4: First suppose there is a branch of $\log(z - z_0)$ on G. Then for all contours Γ on G, we have

$$\int_\Gamma \frac{1}{z - z_0}\, dz = 0$$

by the Fundamental Theorem of Calculus. So then the winding number of Γ around z_0 is 0. Now suppose that z_0 is in a bounded connected component of $\overline{\mathbb{C}} \setminus G$; then, just as in the second half of the proof of the winding number criterion, we can construct a simple contour around this component in G (using the Separation Lemma) that has nonzero winding number around z_0, which contradicts what we have proved above. Thus z_0 is in the unbounded connected component, as we wished.

Now suppose z_0 and ∞ are in the same connected component. Well, then the winding number of any contour Γ in G around z_0 is 0, because this is so in the unbounded component of $\overline{\mathbb{C}} \setminus \Gamma$. So we can define an antiderivative of $\frac{1}{z-z_0}$ as in (X.4); that is, we fix a point $a \in G$ and let $g(z) = c + \int_\gamma \frac{1}{\zeta - z_0}\, d\zeta$ where γ is any path from a to z in G and c is a value of $\log(a - z_0)$. Then g is well-defined etc. etc., just as in (X.4). And, just as in (X.5), an antiderivative of $\frac{1}{z-z_0}$ must be a branch of $\log(z - z_0)$.

Problem 5: Check it out: this is just exercise VIII.4.1 plus the residue theorem! (All the winding numbers of a big enough counterclockwise circle are 1.)

Problem 6: Call the function $f(z)$. Then the integral equals

$$2\pi i(\text{res}_{-1/2}(f) + \text{res}_{1/3}(f)) = -2\pi i\, \text{res}_2(f) = \frac{-2\pi i}{5^5}.$$

Note that the residue at 2 is much easier to compute than the ones at $-1/2$ and $1/3$. That's why the previous exercise is so useful.

Problem 7: Ok, well, it's easy enough to see that the integrals of $1/z^2$ and e^{2iz}/z^2 over S_R are bounded by things that tend to 0 as $R \to \infty$. (The first one is easy, and for the second one, just look at the bounds on p. 123 and use the same inequalities.) So basically I'm telling you that I'm skipping this work. Now the integral over the whole thing is 0, and the limit of the sum of the integrals over the two lines is

$$2\int_0^\infty \frac{1-\cos(2x)}{x^2}\,dx = 4\int_0^\infty \frac{\sin^2(x)}{x^2}\,dx$$

Note that the imaginary part of the integral vanishes because it's an odd function, integrated over a subset of the real axis which is symmetric with respect to the origin. The factor of 2 at the beginning is because the integrand is even.

So Cauchy's theorem implies that our integral equals

$$\lim_{\epsilon\to 0^+} \frac{1}{4}\int_{S_\epsilon} \frac{1-e^{2iz}}{z^2}\,dz = \lim_{\epsilon\to 0^+} \frac{1}{4}\int_{S_\epsilon} \left(-\frac{2i}{z} + f(z)\right)\,dz$$

where $f(z)$ is a holomorphic function. We get this by expanding $1 - e^{2iz}$ out as a Taylor series and dividing by z^2. Now, the integral of a holomorphic function around S_ϵ goes to 0 as $\epsilon \to 0$, because the function is bounded above in a neighborhood of 0 and $L(S_\epsilon) \to 0$. So we get

$$\lim_{\epsilon\to 0^+} \frac{1}{4}\int_{S_\epsilon} -\frac{2i}{z}\,dz = \frac{1}{4}\int_{-\pi}^0 -2i\cdot i\,d\theta = \frac{\pi}{2}.$$

Problem 8: We rewrite it all as an integral around the unit circle γ:

$$\begin{aligned}\int_\gamma \frac{(z+z^{-1})/2}{a-(z+z^{-1})/2}\,\frac{dz}{iz} &= \frac{1}{i}\int_\gamma \frac{(z^2+1)\,dz}{2az^2 - z^3 - z} \\ &= \frac{1}{-i}\int_\gamma \frac{(z^2+1)\,dz}{z(z^2-2az+1)}\end{aligned}$$

Now there is exactly one root of $z^2 - 2az + 1$ inside the unit circle, and that is $b = a - \sqrt{a^2-1}$. Let $f(z)$ be the integrand; then we get

$$\begin{aligned}-2\pi(\text{res}_0 f(z) + \text{res}_b f(z)) &= -2\pi\left(1 + \frac{b^2+1}{b(2b-2a)}\right) \\ &= -2\pi\left(\frac{3b^2 - 2ab + 1}{b(2b-2a)}\right) \\ &= 2\pi\frac{2b^2}{b(2a-2b)} \\ &= \frac{2\pi b}{a-b} \\ &= 2\pi\left(\frac{a}{\sqrt{a^2-1}} - 1\right)\end{aligned}$$