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Math 185, 8/2/05

Solution Set 10

Problem 1: The point here is partial fractions:

$$f(z) = q(z) + \sum_{j=1}^{p} \sum_{k=1}^{m_j} \frac{a_{kj}}{(z-z_j)^k}$$

where $q(z)$ is a polynomial (the quotient after long division of the numerator of f by the denominator), and m_j is the order of the pole at z_j. Now then, note that $q(z)$ and $\frac{a_{kj}}{(z-z_j)^k}$ have antiderivatives in an open set containing γ, unless $k = 1$. So most of the terms in this sum integrate over γ to zero. We get

$$\frac{1}{2\pi i} \int_\gamma f(z)\,dz = \frac{1}{2\pi i} \sum_{j=1}^{p} \int_\gamma \frac{a_{1j}}{z - z_j}\,dz = \sum_{j=1}^{p} a_{1j}\,\mathrm{ind}_\gamma(z_j)$$

But it should be clear that we can read off the principal part of the Laurent series for f around z_j just by looking at the terms of the sum corresponding to z_j (the rest of the terms are holomorphic in a neighborhood of z_j). So of course the residue of f at z_j is the coefficient of $(z - z_j)^{-1}$, which is a_{1j}. So we are done.

Problem 2: Let $z_0 = x_0 + iy_0$. Let's multiply the top and bottom by $t - \overline{z_0}$. Letting $u = t - x_0$, we get

$$\begin{aligned}
\lim_{R\to\infty} \frac{1}{2\pi i} \int_{-R}^{R} \frac{(t-x_0)+iy_0}{(t-x_0)^2+y_0^2}\,dt &= \lim_{R\to\infty} \frac{1}{2\pi i} \left(\int_{-R-x_0}^{R-x_0} \frac{u\,du}{u^2+y_0^2} + i \int_{-R-x_0}^{R-x_0} \frac{y_0\,du}{u^2+y_0^2} \right) \\
&= \lim_{R\to\infty} \frac{1}{2\pi i} \left(\frac{1}{2} \ln(u^2+y_0^2)\Big|_{-R-x_0}^{R-x_0} + i \arctan(u/y_0)\Big|_{-R-x_0}^{R-x_0} \right) \\
&= \lim_{R\to\infty} \frac{1}{2\pi i} \left(\frac{1}{2} \ln \left(\frac{(R-x_0)^2+y_0^2}{(R+x_0)^2+y_0^2} \right) \right) + \frac{1}{2\pi i} \cdot \pi i\,\mathrm{sign}(y_0) \\
&= \frac{1}{2}\,\mathrm{sign}(y_0)
\end{aligned}$$

Problem 3: Fix a point $z_0 \in G$. Let U be the set of points in G that can be joined to z_0 by a polygonal path (which is just a finite number of connected line segments). Let V be the complement of U in G. Clearly U and V are disjoint and their union is all of G. Also, U is nonempty, because there is an open disk around z_0, and (key observation here!) any two points in an open disk can be joined together by a polygonal path.

Now, we'll show that U and V are both open. Suppose $z \in U$. Then take an open neighborhood of z in G. Any point w in this neighborhood can be connected by a polygonal path (just a line segment, in fact) to z. Since $z \in U$, we have a polygonal path from z to z_0, so joining the two paths gives a polygonal path from w to z_0. Thus $w \in U$. So this neighborhood of z lies inside U.

The argument for V is similar. If $z \in V$ and w lies in an open neighborhood of z in G, suppose w can be connected to z_0. Then we can also connect z to z_0, through w. This is a contradiction, so $w \in V$.

Now, by the connectedness assumption, V has to be empty (otherwise we have a decomposition of G as a union of two disjoint nonempty open sets). This is what we wanted.

Remarks: We could specify that the path consist only of line segments parallel to the coordinate axes (same proof). Also, the converse is true (this is not hard to prove).

Problem 4: The idea is to express G as an increasing union of compact sets and apply Runge's theorem. Let

$$K_n = \{z \in G\colon |z| \le n, d(z, \mathbb{C} \setminus G) \ge 1/n\}.$$

Then it is evident that K_n is an increasing sequence of sets whose union is G (check that any element of G must be contained in K_n for sufficiently large n). Plus, each K_n is closed and bounded (easy enough to check), hence compact.

Next note that S contains one point in each connected component of $\overline{\mathbb{C}} \setminus K_n$. Why is this? Well, I claim that each connected component of $\overline{\mathbb{C}} \setminus K_n$ contains a connected component of $\overline{\mathbb{C}} \setminus G$. (Note that this is not just because $\overline{\mathbb{C}} \setminus K_n$ contains $\overline{\mathbb{C}} \setminus G$; for example, a set with two connected components contains one of its components!)

To prove this, first note that the unbounded component of $\overline{\mathbb{C}} \setminus K_n$ certainly contains the unbounded component of $\overline{\mathbb{C}} \setminus G$. Now suppose there is a bounded component U of $\overline{\mathbb{C}} \setminus K_n$ not containing any component of $\overline{\mathbb{C}} \setminus G$. If you think about this (drawing a picture might help), it must follow that U is contained in G! But then, if you look at the definition of K_n, it should follow that all the points in U are contained in K_n too, which is a contradiction. (The picture to have in mind here is of K_n as a closed annulus and U the inner circle. If K_n surrounds a bounded open subset of G, that bounded open subset must also be in K_n.)

Is that enough hand-waving yet? At any rate, we can finally apply Runge's theorem: f is uniformly approximated on K_n by rational functions whose poles lie in S. Let g_n be a rational function whose poles lie in S such that $|f(z) - g_n(z)| < 1/n$ for all $z \in K_n$. Then it should be clear enough that g_n converges locally uniformly to f. This is because any compact subset of G is contained in one of the K_n (why?), so the g_n converge uniformly to f on that subset.

Problem 5: Use the above exercise with $G = \mathbb{C} \setminus \{\operatorname{Re} z = 0\}$ and $S = \{\infty\}$, and $f(z) = \operatorname{sign}(\operatorname{Re} z)$. Then we get a sequence p_k of polynomials (rational functions with poles only at ∞ are polynomials) that does what we want it to on the right and left half planes. (Note also that the convergence is locally uniform, which we will use in a second.) Now, what about the imaginary axis? Let z_0 be a point on the imaginary axis. Then we can use the mean value theorem:

$$p_k(z_0) = \frac{1}{2\pi} \int_0^{2\pi} p_k(z_0 + re^{it})\, dt$$

Because p_k converges locally uniformly to $\operatorname{sign}(\operatorname{Re} z)$, it converges uniformly on the circle of radius r around z_0, so we can use (VI.11) to take $\lim_{k\to\infty}$ inside the integral:

$$\lim_{k\to\infty} p_k(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{sign}(\operatorname{Re}(z_0 + re^{it}))\, dt = 0.$$

This last part of the solution is mostly courtesy of Jed, and is much nicer than my solution.

Problem 6: Let's let R be big enough so that it contains all the zeroes of p and q. Then the quotient rule implies that

$$\frac{f'}{f} = \frac{p'}{p} - \frac{q'}{q}.$$

To integrate both of these fractions around C_R, we use the residue theorem for rational functions. All the winding numbers are 1, and the residue of p'/p at a zero of p is equal to the order of the zero, by exercise VIII.12.3. So then $\frac{1}{2\pi i} \int_{C_R} \frac{p'(z)}{p(z)}\, dz$ equals the sum of the orders of the zeroes of p, which just equals the degree of p (by the Fundamental Theorem of Algebra). Similarly for q, and we're done.