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Math 185, 6/26/05

Solution Set 1

Problem 1: (a) This is true. If we write $z = x + yi$, $w = a + bi$, then

$$\overline{z}\,\overline{w} = (x - yi)(a - bi) = (xa - yb) - (xb + ya)i, \quad zw = (x + yi)(a + bi) = (xa - yb) + (xb + ya)i$$

so the result follows.

(b) This is true, because

$$(|z||w|)^2 = |z|^2|w|^2 = z\overline{z}w\overline{w} = zw(\overline{z}\,\overline{w}) = zw(\overline{zw}) = |zw|^2$$

(note where we used part (a)), and then taking the square root of both sides, and noting that $|z||w|$ and $|zw|$ are both real and nonnegative, the result follows.

(c) This is false. For instance $i \cdot i = -1$, but $0 \cdot 0 \neq -1$.

(d) This is true. Write $z = x + yi$. Then

$$(|x| + |y|)^2 = |x|^2 + 2|x||y| + |y|^2 = x^2 + y^2 + 2|x||y| \geq x^2 + y^2,$$

and then taking square roots of both sides again gives what we want.

Problem 2: First, we compute $(1+i)(239+i) = 238+240i$ and $(5+i)^4 = (24+10i)^2 = 476+480i$. Note that the second number is twice the first. So the two numbers have the same arguments. The argument of $(1 + i)(239 + i)$ is $\pi/4 + \tan^{-1}(1/239)$, and the argument of $(5 + i)^4$ is $4\tan^{-1}(1/5)$. It should be clear that both of these numbers are in the interval $(0, \pi)$, by very easy estimates. So, they must actually be equal (not just equal up to a multiple of 2π). Subtracting $\tan^{-1}(1/239)$ from both sides of this equality gives us what we want.

Remark: This formula is very useful for approximating π, if you use the well-known Taylor series for $\tan^{-1}$.

Problem 3: Let us start with the right side of this equivalence. So:

$$\begin{aligned}
\left|\frac{z-\alpha}{1-\overline{\alpha}z}\right| < 1 \Leftrightarrow |z-\alpha| < |1 - z\overline{\alpha}| &\Leftrightarrow |z-\alpha|^2 < |1 - z\overline{\alpha}|^2 \\
&\Leftrightarrow (z-\alpha)(\overline{z}-\overline{\alpha}) < (1 - z\overline{\alpha})(1 - \overline{z}\alpha) \\
&\Leftrightarrow |z|^2 + |\alpha|^2 - (z\overline{\alpha} + \overline{z}\alpha) < 1 + |z|^2|\alpha|^2 - (z\overline{\alpha} + \overline{z}\alpha) \\
&\Leftrightarrow |z|^2 + |\alpha|^2 - 1 - |z|^2|\alpha|^2 < 0 \\
&\Leftrightarrow (|z|^2 - 1)(1 - |\alpha|^2) < 0 \\
&\Leftrightarrow |z|^2 - 1 < 0 \Leftrightarrow |z| < 1.
\end{aligned}$$

Problem 4: This simply follows from the additivity and multiplicativity of conjugation. Write $P(z) = c_0 + c_1z + \ldots + c_kz^k$, for $c_i \in \mathbb{C}$. Then

$$\begin{aligned}
P(z) = 0 \Leftrightarrow c_0 + c_1z + \ldots + c_kz^k = 0 \Leftrightarrow \overline{c_0 + c_1z + \ldots + c_kz^k} = 0 &\Leftrightarrow \overline{c_0} + \overline{c_1z} + \ldots + \overline{c_kz^k} = 0 \\
&\Leftrightarrow \overline{c_0} + \overline{c_1}\,\overline{z} + \ldots + \overline{c_k}\,\overline{z}^k = 0 \\
&\Leftrightarrow c_0 + c_1\overline{z} + \ldots + c_k\overline{z}^k = 0 \\
&\Leftrightarrow P(\overline{z}) = 0.
\end{aligned}$$

Problem 5: (a) First note that if $Q(z)|R_1(z)$ and $Q(z)|R_2(z)$, then $Q(z)|(c_1R_1(z) + c_2R_2(z))$. (This is obvious.) So then it is enough to show that $(z - z_0)|(z^j - z_0^j)$ for all nonnegative integers

j. Well, for $j = 0, 1$ it is clear, and for $j \geq 2$ one can check directly that

$$(z - z_0)(z^{j-1} + z^{j-2}z_0 + \ldots + zz_0^{j-2} + z_0^{j-1}) = z^j - z_0^j.$$

(b) Let $P(z)$ be the polynomial in question. Base case: degree $P = 1$. No problem here. Now suppose the statement is true for all polynomials of degree $\leq k$. Let $P(z)$ be a polynomial of degree $k+1$. By assumption, $P(z)$ has a root in $\mathbb{C}$, say z_0, and now by part (a) we see that $(z - z_0)|P(z)$. So $(z - z_0)S(z) = P(z)$. Note that the degree of S is k, so S factors into linear factors, and therefore P does as well. So we have proved the statement by induction.

Problem 6: This is pretty easy, since we already found the roots of unity in class. Note that the set of solutions to $z^n = 1$ is precisely $T = \{1, \zeta, \ldots, \zeta^{n-1}\}$ (check your notes, and de Moivre).

First, if $a = 0$, then it should be clear that the only root of $z^n - a$ is 0. (Why?) But in this case, $b = 0$, so $\{b, \zeta b, \ldots, \zeta^{n-1}b\} = \{0\}$, so the statement is ok in this case.

Now then, if $a \neq 0$, $b \neq 0$, so:

$$z^n = a \Leftrightarrow (z/b)^n = a/a = 1 \Leftrightarrow \frac{z}{b} \in T \Leftrightarrow z \in \{b, \zeta b, \ldots, \zeta^{n-1}b\}.$$

Problem 7: We have

$$(-\sqrt{3} + i)^{103} = (2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}))^{103} = 2^{103}(\cos\frac{206\pi}{3} + i\sin\frac{206\pi}{3}) = 2^{103}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$$
$$= 2^{102}(-\sqrt{3} + i).$$

Problem 8: First of all, I claim that z_1, z_2, z_3 are the vertices of an equilateral triangle if and only if $z_3 - z_1 = \omega(z_2 - z_1)$, where $\omega = \cos(\pi/3) + i\sin(\pi/3) = 1/2 + i\sqrt{3}/2$ or its conjugate. Geometrically, this is because the above equality is true if and only if $|z_3 - z_1| = |z_2 - z_1|$ (because $|\omega| = 1$) and the angle at z_1 is 60^o. (Why?) This is equivalent to saying that the triangle is isoceles, so two angles are equal, and also that the third angle is 60^o. So this is indeed equivalent to the triangle being equilateral.

You can then solve the equation given in the problem for z_3, using the "quadratic formula" (being careful to make sense out of square roots of complex numbers), to show that the two solutions to it are $z_1 + \omega(z_2 - z_1)$, where ω is as above. This would be fine. Let me now show you a slightly more elegant proof.

I claim instead that z_1, z_2, z_3 are the vertices of an equilateral triangle if and only if

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{z_1 - z_2}{z_3 - z_2} \tag{1}$$

Why is this? Well, first of all, the arguments of the two fractions are exactly the angles at z_1 and z_2, respectively. (Why? Note that there is a sign issue here that you must resolve.) Now (1) is true if and only if the angles at z_1 and z_2 are equal and the absolute values of both sides of (1) are equal. But the angles at z_1 and z_2 are equal if and only if $|z_3 - z_1| = |z_3 - z_2|$. So

$$|z_3 - z_1||z_3 - z_2| = |z_1 - z_2|^2 \Leftrightarrow |z_3 - z_2| = |z_1 - z_2|$$

and these two equalities establish that the triangle is equilateral. (The reverse direction, showing that (1) holds if the triangle is equilateral, is similar and easier.)

It remains only to note that equation (1) can be multiplied out to give

$$(z_3 - z_1)(z_3 - z_2) = (z_2 - z_1)(z_1 - z_2) \Leftrightarrow z_3^2 - z_3z_1 - z_2z_3 + z_1z_2 = 2z_1z_2 - z_1^2 - z_2^2$$
$$\Leftrightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

and so we are done.

Problem 9: Write $z = x + yi$. Then the equation turns into

$$\alpha(x^2 + y^2) + \beta x + \gamma y + \delta = 0$$

so if $\alpha = 0$, we clearly have a line. If α is nonzero, we divide out and complete the square to get

$$\left(x + \frac{\beta}{2\alpha}\right)^2 + \left(y + \frac{\gamma}{2\alpha}\right)^2 = \frac{\beta^2 + \gamma^2}{4\alpha^2} - \frac{\delta}{\alpha}$$

which is clearly either empty, a point, or a circle, depending on whether the right side is negative, zero, or positive.

For the converse, note that any line is of the form $\beta x + \gamma y + \delta = 0$, so it's immediate that it can be described by this equation (just take $\alpha = 0$). Now consider a circle, with equation $(x - a)^2 + (y - b)^2 = c^2$. Multiplying out and grouping terms, we get

$$(x^2 + y^2) - 2ax - 2by + (a^2 + b^2 - c^2) = 0$$

which is in the appropriate form.

Problem 10: Let $\theta = \cos^{-1} x$. Then de Moivre's formula gives

$$\cos n\theta + i \sin n\theta = \sum_{j=0}^{n} \binom{n}{j} i^j \cos^{n-j}\theta \sin^j \theta$$

and we can compare real parts. The terms of the sum on the left are either purely real or purely imaginary, depending on whether j is even or odd, respectively. For even j, write $j = 2k$. Note $i^j = (-1)^k$. Let $m = \lfloor n/2 \rfloor$ be the integer part of $n/2$. We get

$$\cos n\theta = \sum_{k=0}^{m} (-1)^k \binom{n}{2k} \cos^{n-2k}\theta \sin^{2k}\theta.$$

Then plugging in $x = \cos\theta$ gives

$$T_n(x) = \sum_{k=0}^{m} (-1)^k \binom{n}{2k} x^{n-2k}(1 - x^2)^k$$

and we can easily write down the first five Chebyshev polynomials:

$$\begin{aligned}
T_1(x) &= x \\
T_2(x) &= x^2 - (1 - x^2) = 2x^2 - 1 \\
T_3(x) &= x^3 - 3x(1 - x^2) = 4x^3 - 3x \\
T_4(x) &= x^4 - 6x^2(1 - x^2) + (1 - x^2)^2 = 8x^4 - 8x^2 + 1 \\
T_5(x) &= x^5 - 10x^3(1 - x^2) + 5x(1 - x^2)^2 = 16x^5 - 20x^3 + 5x
\end{aligned}$$

Remark: You might try to prove that $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$; the best way would probably be to use appropriate formulas from trigonometry, but you might also consider induction, or maybe directly using the formula if you love long computations.