

Midterm 2

Problem 1: Let r be a positive real number, and let $D_r = \{z \in \mathbb{C} : |z| < r\}$. Suppose $f: D_r \rightarrow \mathbb{C}$ is holomorphic, with $f^{(n)}(0) = n^2 \cdot n!$ for all nonnegative integers n .

(a) (10 points) How big can r be?

(b) (15 points) Show that f is a rational function, and find f . (Hint: Start with $\frac{1}{1-z}$, and do some differentiating and some other things.)

Problem 2: (20 points) Recall that $\sin k\theta = \frac{1}{2i}(z^k - z^{-k})$, where $z = e^{i\theta}$ and $k \in \mathbb{Z}$. (Just a reminder—you don't have to prove it, but you'll probably want to use it!)

Now, let n be a positive integer. Show that

$$\int_{-\pi}^{\pi} \frac{\sin n\theta}{\sin \theta} d\theta = \begin{cases} 2\pi & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

(Hint: Parameterize the unit circle by $z = e^{i\theta}$, $-\pi \leq \theta \leq \pi$. Turn this into an integral around the unit circle; you'll probably want to explain why $d\theta = \frac{dz}{iz}$.)

Problem 3: Suppose $f: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ is a holomorphic function satisfying the inequality

$$|f(z)| \geq \frac{1}{\sqrt{|z|}}$$

for all $z \in \mathbb{C} \setminus \{0\}$.

(a) (5 points) Show that f has a pole at 0.

(b) (5 points) Explain why there is some $m \in \mathbb{N}$ such that $h(z) = z^m f(z)$ can be extended to an entire function such that $h(0) \neq 0$.

(c) (5 points) Show that h has a pole at ∞ .

(d) (15 points) Show that f cannot exist! (Hint: Can h have any zeroes? What kind of function is h ?)

Problem 4: Let $f(z) = e^{1/z}$. Let γ be the unit circle oriented counterclockwise.

(a) (5 points) Find the Laurent series for $f(z)$ in the punctured disk $\mathbb{C} \setminus \{0\}$. What kind of singularity is 0?

(b) (20 points) As we have seen, we can recover the coefficients of this Laurent series $\sum_{k=-\infty}^{\infty} a_k z^k$ in another way, by integrating a suitable function around γ . Use the value of a_{-n} you found in part (a) to show that

$$\int_{-\pi}^{\pi} e^{\cos \theta} \cos(\sin \theta - n\theta) d\theta = \frac{2\pi}{n!}$$

for all nonnegative integers n . What do you get if n is negative?