Patrick Corn
Math 185
7/7/05

Midterm 1

Problem 1: (a) (10 points) Suppose a and b are two nonzero complex numbers. Show that

$$a + \frac{1}{a} = b + \frac{1}{b}$$

if and only if $a = b$ or $a = 1/b$. (Hint: Multiply both sides of the equation by ab.)

(b) (15 points) Suppose z and w are two complex numbers. Show that $\cos z = \cos w$ if and only if $z = \pm w + 2\pi n$, for some choice of the sign and some $n \in \mathbb{Z}$.

Problem 2: For $a_1, a_2, \ldots, a_n \in \mathbb{C}$, let $P(z) = (z - a_1)(z - a_2) \cdots (z - a_n)$.
(a) (5 points) Show that if $z \neq a_i$,

$$\frac{P'(z)}{P(z)} = \sum_{i=1}^{n} \frac{1}{z - a_i}$$

(b) (5 points) For $w \in \mathbb{C}$, show that if $\operatorname{Re} w > 0$, then $\operatorname{Re}(1/w) > 0$.

(c) (15 points) Suppose that $\operatorname{Re} a_i < 0$ for all i. Suppose $P'(b) = 0$. Show that $\operatorname{Re} b < 0$. (Hint: Proof by contradiction.)

Problem 3: Let $\phi(z) = \frac{z-1}{z-i}$. Let γ_1 be the line $\operatorname{Im} z = 1$. Let γ_2 be the line $\operatorname{Re} z = 1$. Let γ_3 be the circle $|z| = 1$.

(a) (15 points) Describe $\phi(\gamma_i)$ for $i = 1, 2, 3$. Pictures are ok, but I also want explicit algebraic descriptions.

(b) (15 points) Let $H = \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}$. Find the subset G of $\overline{\mathbb{C}}$ that satisfies $\phi(G) = H$.

Problem 4: Surprise! The following four true-false questions are worth 5 points each. Just write TRUE or FALSE (not just T or F!) for each question; I will not look at justifications of your answers. (In other words: no partial credit.)

(a) The function $f(z) = |z|^2$ is not differentiable (in the complex sense) at any point in $\mathbb{C}$.

(b) If there is a branch of $\arg z$ in an open connected set $N \subseteq \mathbb{C}$, and $0 \notin N$, then there is a branch of $\sqrt{z}$ in N.

(c) The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = \mathrm{Re}(e^{(x+yi)^2} \sin(x+yi))$ is harmonic everywhere.

(d) If $f\colon G \to \mathbb{C}$ is conformal on G, then it is one-to-one.