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Math 185, summer 2005

**Homework 1**
**Due 6/23/05**

**Problem 1:** For each of the following statements, give a proof or a counterexample.
(a) $\overline{z}\,\overline{w} = \overline{zw}$
(b) $|z||w| = |zw|$
(c) $(\operatorname{Re} z)(\operatorname{Re} w) = \operatorname{Re} zw$
(d) $|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$

**Problem 2:** Show that
$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$
(hint: compare $(1+i)(239+i)$ and $(5+i)^4$).

**Problem 3:** Consider $\alpha \in \mathbb{C}, |\alpha| < 1$. Show that
$$|z| < 1 \text{ if and only if } \left|\frac{z-\alpha}{1-\overline{\alpha}z}\right| < 1.$$

**Problem 4:** Let $P$ be a polynomial with real coefficients. Show that $P(z) = 0$ if and only if $P(\overline{z}) = 0$.

**Problem 5:** If $Q(z)$ and $R(z)$ are polynomials, then we say that $Q(z)|R(z)$ if and only if there is a polynomial $S(z)$ such that $Q(z)S(z) = R(z)$.

(a) For any polynomial $P(z)$ with complex coefficients, and any $z_0 \in \mathbb{C}$, show that $(z-z_0)|(P(z)-P(z_0))$. (Hint: It may help to write $P(z) = \sum_{j=0}^{k} c_j z^j$, for some coefficients $c_j \in \mathbb{C}$.)

(b) Suppose that every non-constant polynomial with complex coefficients has a root in $\mathbb{C}$. (Later, we will show that this is in fact true.) Prove that every non-constant polynomial with complex coefficients factors over $\mathbb{C}$ as a product of linear factors. (Hint: Induction on the degree of the polynomial.)

**Problem 6:** Fix an integer $n \geq 1$, and let $\zeta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Suppose we are given $a, b \in \mathbb{C}$ such that $b^n = a$. Then show that the set of roots of the polynomial $z^n - a$ is
$$\{b, \zeta b, \ldots, \zeta^{n-1}b\}.$$

**Problem 7:** Compute $(-\sqrt{3}+i)^{103}$ without a calculator. Show your work!

**Problem 8:** Show that three points $z_1, z_2, z_3 \in \mathbb{C}$ are the vertices of an equilateral triangle in the complex plane if and only if
$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$
(Hint: You might try simplifying the problem by first translating, dilating, and rotating. Can you do it if $z_1 = 0$ and $z_2 = 1$?)

**Problem 9:** Consider the set of points $z \in \mathbb{C}$ satisfying the equation
$$\alpha|z|^2 + \beta\operatorname{Re} z + \gamma\operatorname{Im} z + \delta = 0$$
where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are constants. Show that this set is either a line, a circle, a point, or the empty set. And conversely, show that any line or circle can be described by this equation, for some choices of $\alpha, \beta, \gamma, \delta$.

**Problem 10:** Expand the left side of de Moivre's formula
$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

to show that the functions $T_n(x) = \cos(n \cos^{-1} x)$ are in fact polynomials in $x$. Find $T_i(x), 1 \leq i \leq 5$. (The $T_i$ are called Chebyshev polynomials.)