

# Stable Reduction of $X_0(p^3)$

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October 13, 2009

## Abstract

We determine the stable models of the modular curves  $X_0(p^3)$  for primes  $p \geq 13$ . An essential ingredient is the close relationship between the deformation theories of elliptic curves and formal groups, which was established in the Woods Hole notes of 1964. This enables us to apply results of Gross-Hopkins in our analysis of the supersingular locus.

## 1 Introduction

Let  $n$  be an integer and  $p$  a prime. It is known that if  $n \geq 3$  and  $p \geq 5$ , or if  $n \geq 1$  and  $p \geq 11$ , the modular curve  $X_0(p^n)$  does not have a model with good reduction over the ring of integers of any complete subfield of  $\mathbb{C}_p$ . By a model for a scheme  $C$  over a complete local field  $K$ , we mean a scheme  $S$  over the ring of integers  $\mathcal{O}_K$  of  $K$  such that  $C \cong S \otimes_{\mathcal{O}_K} K$ . When a curve  $C$  over  $K$  does not have a model with good reduction over  $\mathcal{O}_K$ , it may have the “next best thing,” i.e., a stable model. The stable model is unique up to isomorphism if it exists, and it does over the ring of integers in some finite extension of  $K$ , as long as the genus of the curve is at least 2. Moreover, if  $\mathcal{C}$  is a stable model for  $C$  over  $\mathcal{O}_K$ , and  $K \subseteq L \subseteq \mathbb{C}_p$ , then  $\mathcal{C} \otimes_{\mathcal{O}_K} \mathcal{O}_L$  is a stable model for  $C \otimes_K L$  over  $\mathcal{O}_L$ . The special fiber of any stable model for  $C$  is called the stable reduction.

The following is a brief summary of prior results regarding the stable models of modular curves at prime power levels. Deligne and Rapoport [DR, §VI.6] found models for  $X_0(p)$  and  $X_1(p)$  over  $\mathbb{Z}_p$  and  $\mathbb{Z}_p[\mu_p]$  which become stable over the quadratic unramified extension. Edixhoven [E, Thm 2.1.2] found stable models for  $X_0(p^2)$  over the ring of integers,  $R$ , in the Galois extension of  $\mathbb{Q}_p^{unr}$  of degree  $(p^2 - 1)/2$ . Bouw and Wewers found stable models of  $X_0(p)$  and  $X(p)$  over  $\mathbb{Z}_p$  and  $R$  by completely different means in [BW, Thm 4.1 and Cor 3.4]. Krir [Kr, Théorème 1] proved that the Jacobian of  $X_0(p^n)$  has a semi-stable model over the ring of integers of an explicit Galois extension  $L_n$  of  $\mathbb{Q}_p^{unr}$  of degree  $p^{2(n-2)}(p^2 - 1)$  for  $n \geq 2$ , which implies that  $X_0(p^n)$  has a stable model over the ring of integers of  $L_n$  by Theorem 2.4 of [DM]. Also, stable models for  $X_0(125)$  and  $X_0(81)$  were computed explicitly in [M1, §2] and [M2, §3] (resp), and a conjectural stable reduction of  $X_0(p^4)$  given in [M2, §5]. The main result of this paper is the construction of a stable model for  $X_0(p^3)$ , when  $p \geq 13$ ,

over the ring of integers of some finite extension of  $\mathbb{Q}_p$  which is made explicit in [CMc].

We introduce the notion of a semi-stable covering of a smooth complete curve over a complete non-archimedean field in Section 2.3. We prove that any curve over a complete stable subfield of  $\mathbb{C}_p$  has a semi-stable covering if and only if it has a semi-stable model, and moreover we can determine the corresponding reduction from the covering (see Theorem 2.36). Finding a semi-stable covering is often easier in practice than finding a semi-stable model directly, and this is what we do for  $X_0(p^3)$  in Sections 6-9.

## 1.1 Overview of Paper

The approach of this paper is rigid analytic, in the sense that we construct a stable model of  $X_0(p^3)$  by actually constructing a stable covering by wide opens (an equivalent rigid analytic notion introduced in [CM, §1]). A covering  $\mathcal{C}^o$  of the ordinary locus can be obtained by extending the ordinary affinoids  $\mathbf{X}_{ab}^\pm$  defined in [C3, §1] to wide open neighborhoods  $W_{ab}^\pm$ . The supersingular locus essentially breaks up into the union of finitely many deformation spaces of height 2 formal groups with level structure (by [WH]). We use results from [GH] and [dSh] to produce a covering  $\mathcal{C}^s$  of this region. Finally, we show that the “genus” of the covering,  $\mathcal{C}^o \cup \mathcal{C}^s$ , is at least the genus of  $X_0(p^3)$ , and therefore that the overall covering is stable. Section-by-section, this argument is laid out as follows.

First, in Section 2, we recall or prove the general rigid analytic results which are necessary for a stable covering argument. These results are proven not only over complete subfields of  $\mathbb{C}_p$ , but over more general complete non-archimedean valued fields. For example, Proposition 2.34 is the aforementioned result that the genus of any stable covering must equal the genus of the curve. We also revise and extend results of Bosch, and of Bosch and Lütkebohmert, on the rigid geometry of algebraic curves. A rigid analytic version of the Riemann Existence Theorem is proven in Appendix A.

In Section 3, we recall and fix notation for some results specifically pertaining to  $X_0(p^n)$  and its rigid subspaces. This is done from the moduli-theoretic point of view, which is that points of  $X_0(p^n)$  correspond to pairs  $(E, C)$  where  $E$  is a generalized elliptic curve and  $C$  is a cyclic subgroup of order  $p^n$ . For example, there is a detailed discussion in Section 3.1 of the theory of the canonical subgroup of an elliptic curve and its connection with the geometry of  $X_0(p)$ , for which [Bu, §3] is our primary reference. Also, Section 3.2 is where we define wide open neighborhoods,  $W_{ab}^\pm \supseteq \mathbf{X}_{ab}^\pm$ , of the irreducible affinoids which make up the ordinary locus of  $X_0(p^n)$ .

All of the results which we need regarding deformations of formal groups are given in Section 4. First we give a precise statement (from [WH, §6]) of the relationship between deformations of elliptic curves and formal groups, which is what we call “Woods Hole Theory.” This is then used in Section 4.1 (along with the result of E. Howe in Appendix B) to prove that all of the connected components of the supersingular locus of  $X_0(p^n)$  are (nearly) isomorphic. Because

of this fact, we are able to focus on those regions,  $W_A(p^n)$ , which correspond to a supersingular elliptic curve  $A/\mathbb{F}_p$  for which  $j(A) \neq 0, 1728$ . Specifically, this enables us to directly apply results of de Shalit ([dSh, §3]) for the forgetful map from  $X_0(p)$  to the  $j$ -line. The other important consequence of Woods Hole Theory is that it gives us a natural action of  $\text{Aut}(\hat{A}) \cong (\text{End}(A) \otimes \mathbb{Z}_p)^*$  on  $W_A(p^n)$ . In Section 4.2 we recall results of Gross-Hopkins which describe this action in great detail, and we derive the specific consequences which we need for our analysis of  $X_0(p^3)$ .

Once the groundwork has been done, the remaining sections are devoted to constructing stable coverings of  $X_0(p^2)$  and  $X_0(p^3)$ . In Section 5 we construct a stable covering for  $X_0(p^2)$  over an explicit Galois extension of  $\mathbb{Q}_p$  of degree  $12(p^2 - 1)$ , essentially showing that the wide open subspaces defined in Section 3 are sufficient. To be more precise, the stable covering consists of

$$\{W_{20}, W_{11}^+, W_{11}^-, W_{02}\} \cup \{W_A(p^2) : A \text{ supersingular}\}.$$

This reproves Edixhoven's result from the point of view of this paper. It also gives a moduli-theoretic interpretation to the wide opens and underlying affinoids in the stable covering.

As in the stable covering of  $X_0(p^2)$ , the ordinary region of  $X_0(p^3)$  is covered by six wide opens:  $W_{30}$ ,  $W_{21}^\pm$ ,  $W_{12}^\pm$ , and  $W_{03}$ . Unlike  $W_A(p^2)$ , however,  $W_A(p^3)$  must itself be covered by smaller wide opens, since its reduction contains multiple irreducible components. First of all, the reduction of  $W_A(p^3)$  contains two isomorphic lifts of some supersingular component of  $X_0(p^2)$ , with each meeting exactly three of the ordinary components. These two "old" components are connected through a central genus 0 component which we call the "bridging component." To complete the picture, the bridging component then meets (in distinct points) a certain number of isomorphic copies of the curve,  $y^2 = x^p - x$ . A partial picture of the stable reduction of  $X_0(p^3)$ , including one complete supersingular region (corresponding to a fixed supersingular curve  $A$ ) and the six ordinary components, is given below in Figure 1. The number and genera of the components, as marked on the graph, are as follows:

$$(a, b, c) = \begin{cases} (\frac{p-1}{2}, \frac{2(p+1)}{3}, \frac{p-5}{6}) & \text{if } j(A) = 0 \\ (\frac{p-1}{2}, p+1, \frac{p-3}{4}) & \text{if } j(A) = 1728 \\ (\frac{p-1}{2}, 2(p+1), \frac{p-1}{2}) & \text{otherwise.} \end{cases}$$

Complete graphs with intersection multiplicities are given in Section 9.1. As a consequence of these results, it follows that the new part of  $J_0(p^3)$  has potential good reduction isogenous to the product of  $(p^2 - 1)/6$  copies of the Jacobian of  $y^2 = x^p - x$ .

It should be noted that the field of definition of our stable covering ultimately depends on the field of definition of certain elliptic curves which have "Fake CM." In [CMc] we will prove results about these Fake CM curves which make it possible, then, to define our stable model over the ring of integers of an

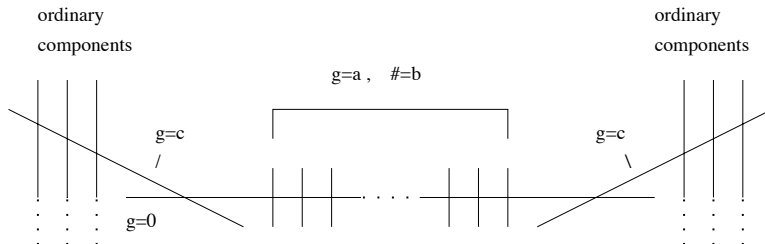


Figure 1: Partial Picture of the Stable Reduction of  $X_0(p^3)$

explicit finite extension of  $\mathbb{Q}_p$  (and compute the associated Weil group action). In [CMc] we will also deal with the  $p \leq 11$  cases explicitly and add tame level, i.e. compute the stable reduction of  $X_0(Np^3)$  for  $(N, p) = 1$ . We expect that our methods will extend to  $X_0(Np^n)$ , and will have applications to modular forms as in [CMc, Rmk. 6.10]. We understand that Wewers also has a different approach with applications to local Langlands.

## 1.2 Acknowledgments

We are grateful to Ken Ribet for explaining to us how potential good reduction of  $J_0(p^3)^{\text{new}}$  follows from known results. This greatly simplified our search for the stable models. In Theorem 9.4 we show how the generalization of this result for  $J_0(p^n)^{\text{new}}$  (which follows from work of Katz-Mazur) can be used to compute the toric rank of  $J_0(p^n)$ . In addition to Ken Ribet, we would also like to express our appreciation to Kevin Buzzard, Brian Conrad, Dino Lorenzini, and Jonathan Lubin for helpful communications. The referee also made a number of suggestions which led to significant improvements in the manuscript, particularly in Section 2.

## 2 Rigid Analytic Foundation

For starters, we fix some notation for the  $p$ -adic analysis and more general non-archimedean analysis which follows in Section 2.

Throughout this section, unless otherwise stated, we let  $K$  be a complete non-archimedean valued field with absolute value  $|\cdot|$ . We denote the ring of integers of  $K$  by  $R_K$ , its maximal ideal by  $m_K$ , and the residue field by  $\mathbb{F}_K$ . Let  $p$  be the characteristic of  $\mathbb{F}_K$  (which we allow to be 0). Fix  $\mathbf{C}$  to be the completion of an algebraic closure of  $K$ , and denote its ring of integers, maximal ideal, and residue field by  $\mathbf{R}$ ,  $m_{\mathbf{R}}$ , and  $\overline{\mathbb{F}}$ , respectively. Note that  $\overline{\mathbb{F}}$  is then an algebraic closure of  $\mathbb{F}_K$ . Whenever  $\mathbb{F}_K$  is perfect and has positive characteristic, we let  $W(\mathbb{F}) \subseteq \mathbf{R}$  denote the ring of Witt vectors for any field  $\mathbb{F} \subseteq \overline{\mathbb{F}}$ . The value group of  $K$  will be denoted  $|K^*|$ , and we let  $\mathcal{R} := \mathcal{R}_K = \{x \in \mathbb{R} :$

$x^n \in |K^*|$  for some  $n \in \mathbb{N}$  (equivalently,  $\mathcal{R} := |\mathbf{C}^*|$ ). Then if  $S \subseteq \mathbb{R}$  we let  $\mathcal{R}S = \mathcal{R} \cap S$ .

Occasionally, for technical reasons, we will need to assume that  $K$  is a stable field (see [BGR, Def. 3.6.1/1]). By [BGR, Prop. 3.6.2/6], this is the case if and only if  $e(L/K)f(L/K) = [L : K]$  for all finite extensions  $L/K$ , where  $e(L/K) = |L^*|/|K^*|$  and  $f(L/K) = [\mathbb{F}_L : \mathbb{F}_K]$  are the ramification index and residue degree of  $L$  over  $K$ , respectively. There are also two special cases which will be considered for certain results. First, for a fixed prime  $p$ , we let  $\mathbb{C}_p$  be the completion of a fixed algebraic closure of  $\mathbb{Q}_p$ ,  $\mathbb{R}_p \subseteq \mathbb{C}_p$  its ring of integers, and  $m_{\mathbb{R}_p}$  the maximal ideal of  $\mathbb{R}_p$ . Let  $v$  denote the unique valuation on  $\mathbb{C}_p$  with  $v(p) = 1$ , and  $|\cdot|$  the absolute value given by  $|0| = 0$  and  $|x| = p^{-v(x)}$  for  $x \neq 0$ . Note that in this case  $\mathcal{R} = |\mathbf{C}_p^*| = p^{\mathbb{Q}}$ . We also note that  $\mathbb{C}_p$  is stable, as is the completion of any tamely ramified extension of a finite extension of  $\mathbb{Q}_p$ . The second specific non-archimedean valued field which will be considered is  $\overline{\mathbb{F}}_p((T))$ , for which the corresponding field  $\mathbf{C}$  will be denoted  $\Omega_p$ . Both  $\overline{\mathbb{F}}_p((T))$  and  $\Omega_p$  are stable, and in this case we have  $\mathcal{R} = |T|^{\mathbb{Q}}$ . We say that  $K$  satisfies **Hypothesis T** if  $\mathbf{C}$  is isomorphic to either  $\mathbb{C}_p$  or  $\Omega_p$ .

**Remark 2.1.** *Suppose  $K$  satisfies Hypothesis T. Then if  $A$  is an Abelian variety over  $K$  and  $P \in A(\mathbf{C})$ ,  $0$  is in the closure of  $\{nP : n \in \mathbb{N}\}$ . (See the proof of Lemma 2.19.)*

Now, for  $r \in \mathcal{R}$ , we let  $B_K^d[r]$  and  $B_K^d(r)$  denote the closed and open  $d$ -dimensional polydisks over  $K$  of radius  $r$  around 0, i.e., the rigid spaces over  $K$  whose  $\mathbf{C}$ -valued points are  $\{(x_1, \dots, x_d) \in \mathbf{C}^d : |x_i| \leq r\}$  and  $\{(x_1, \dots, x_d) \in \mathbf{C}^d : |x_i| < r\}$ , respectively. In particular, let  $B_K[r] := B_K^1[r]$  and  $B_K(r) := B_K^1(r)$  denote the **closed disk** and **open disk** of radius  $r$  around 0. If  $r, s \in \mathcal{R}$  and  $r \leq s$ , let  $A_K[r, s]$  and  $A_K(r, s)$  be the rigid spaces over  $K$  whose  $\mathbf{C}$  valued points are  $\{x \in \mathbf{C} : r \leq |x| \leq s\}$  and  $\{x \in \mathbf{C} : r < |x| < s\}$ , which we call **closed annuli** and **open annuli**. The **semi-open annuli**,  $A_K[r, s)$  and  $A_K(r, s]$ , are similarly defined. The width of such an annulus is defined to be  $\log_p(s/r)$  or  $\ln(s/r)$  if  $p = 0$ . Note that all closed or open disks over  $K$ , and all closed or open annuli over  $K$  of the same width, are potentially isomorphic. Here and throughout the paper, we use the adverb ‘‘potentially’’ in various contexts to mean ‘‘after finite base extension.’’ A closed annulus of width 0 will be called a **circle**, and we will also denote the circle,  $A_K[s, s]$ , by  $C_K[s]$ .

If  $X$  is a rigid space over  $K$  and  $f \in A(X) := \mathcal{O}_X(X)$ , let  $|f|_{sup}$  denote the sup of  $|f(x)|$  over all  $x \in X(\mathbf{C})$ . Then set

$$\begin{aligned} A^o(X) &= \{f \in A(X) : |f|_{sup} \leq 1\} \\ A^+(X) &= \text{cl} \{f \in A(X) : |f|_{sup} < 1\} \\ \overline{A(X)} &= A^o(X)/A^+(X), \end{aligned}$$

where  $\text{cl}$  is the closure in  $A^o(X)$ . We define the reduction of  $X$ , denoted  $\overline{X}$ , to be the affine scheme,  $\text{Spec } \overline{A(X)}$ . Suppose now that  $X = \text{Sp}(A)$  is an affinoid. Then  $|f|_{sup}$  is just the usual spectral semi-norm of  $f$ , which we also denote by

$\|f\|_X$  when  $X$  is reduced and  $|\cdot|_{sup}$  is a norm. There is a canonical reduction map,  $\text{Red} : X(\mathbf{C}) \rightarrow \overline{X}(\mathbb{F})$ , which we denote by  $x \mapsto \bar{x}$ . If  $X$  is reduced and  $\tilde{Y}$  is any subscheme of  $\overline{X}$ , then  $Y := \text{Red}^{-1}\tilde{Y}$  is the rigid space admissibly covered by affinoid subdomains  $Z$  of  $X$  such that  $\overline{Z}$  maps into  $\tilde{Y}$ . As a special case, when  $\tilde{Y} \subseteq \overline{X}$  is an open affine,  $Y$  is the unique subaffinoid of  $X$  such that  $Y(\mathbf{C}) = \{x \in X(\mathbf{C}) : \bar{x} \in \tilde{Y}(\mathbb{F})\}$ , and we call  $Y$  a Zariski subaffinoid of  $X$ . When  $\overline{X}$  is a reduced affine curve, we let  $\overline{X}^c$  denote the unique complete curve which contains  $\overline{X}$  as an affine open and is nonsingular at all other points (which we call the “points at infinity”).

If  $X$  is a rigid space over  $K$ , and  $L \supseteq K$  is a complete subfield of  $\mathbf{C}$ , we write  $P \in X(L)$  to mean that  $P$  is an  $L$ -valued point of  $X$ . An unspecified  $P \in X$  should be read as  $P \in X(\mathbf{C})$ . We use the notations  $X_L$  and  $X_{\mathbb{F}_L}$  for the extensions of  $X$  and  $\overline{X}$  by scalars, respectively.

## 2.1 Annuli

**Proposition 2.2.** *Let  $f : \mathcal{A}_1 \rightarrow \mathcal{A}_2$  be a degree  $d$  unramified surjection of annuli over  $K$  (open or closed). Then the width of  $\mathcal{A}_1$  is  $1/d$  times the width of  $\mathcal{A}_2$ .*

*Proof.* Extend scalars to  $\mathbf{C}$ , and choose isomorphisms,  $\psi_i : \mathcal{A}_i \rightarrow A_{\mathbf{C}}(r_i, 1)$ , for some  $r_i \in \mathcal{R}$  with  $r_i < 1$ . Let  $T$  be the natural parameter on  $A_{\mathbf{C}}(r_1, 1)$ . Then viewing  $\tilde{f} := \psi_2 \circ f \circ \psi_1^{-1}$  as an invertible function on  $A_{\mathbf{C}}(r_1, 1)$ , we may write  $\tilde{f}$  as either  $cT^d(1+g(T))$  or  $cT^{-d}(1+g(T))$ , where  $g(T) \in A^+(A_{\mathbf{C}}(r_1, 1))$ . In the first case, for any  $t \in A_{\mathbf{C}}(r_1, 1)$ , we clearly have  $|\tilde{f}(t)| = |c| \cdot |t|^d$ . So by surjectivity of  $f$ , this implies that  $|c| = 1$  and  $r_1^d = r_2$ . Thus,  $\log_p(1/r_1) = \frac{1}{d} \log_p(1/r_2)$ . The second case is very similar.  $\square$

**Definition 2.3.** *For any  $r \in \mathbb{R}_+^* \setminus \mathcal{R}$ , we let*

$$K_r = \left\{ \sum_{n \in \mathbb{Z}} a_n T^n : a_n \in K, \lim_{|n| \rightarrow \infty} |a_n| r^n = 0 \right\}.$$

*Then  $K_r$  is a field, and if  $f(T) = \sum_{n \in \mathbb{Z}} a_n T^n$ ,  $f \mapsto \max\{|a_n| r^n\}$  is a valuation.<sup>1</sup> If  $r_1, \dots, r_n \in \mathbb{R}_+^*$  have linearly independent images in the  $\mathbb{Q}$ -vector space,  $\mathbb{R}_+^*/\mathcal{R}$ , we let  $K_{r_1, \dots, r_n} := (K_{r_1, \dots, r_{n-1}})_{r_n}$  and  $K_\emptyset = K$ .*

Then  $K_{r_1, \dots, r_n} \cong K_{r_1} \hat{\otimes}_K \cdots \hat{\otimes}_K K_{r_n}$ , and its value group is generated by  $\mathcal{R}$  and  $\{r_1, \dots, r_n\}$  [Be, pg. 21–22]. If  $m$  is a positive integer, the map  $f(T) \mapsto f(T^m)$  gives an injection from  $K_{r^m}$  into  $K_r$ , for any  $r$ .

**Lemma 2.4.** *The group  $\text{Aut}_{\text{cont}}(K_{r_1, \dots, r_n}/K)$  contains a subgroup  $H$ , for which  $h \mapsto h^d$  is a bijection whenever  $p \nmid d$ , and whose fixed field is  $K$ .*

*Proof.* It suffices to do the case  $n = 1$ . Let  $r = r_1$ . Suppose  $\alpha \in K$  such that  $|\alpha| < r$ . If  $f \in K_r$ ,  $f(T) = \sum_{n \in \mathbb{Z}} a_n T^n$ , we set

$$f^{\sigma_\alpha}(T) = \sum_{n > 0} a_{-n} \left( \frac{T^{-1}}{1 - \alpha T^{-1}} \right)^n + \sum_{n \geq 0} a_n (T - \alpha)^n.$$

<sup>1</sup>Some authors call this an absolute value.

Then  $\sigma_\alpha \in \text{Aut}_{\text{cont}}(K_r/K)$ , and if  $a_n = 0$  for large  $|n|$ ,  $f^{\sigma_\alpha}(T)$  is the image of the rational function  $f(T-\alpha)$  in  $K_r$ . It follows, by continuity, that the map,  $\alpha \mapsto \sigma_\alpha$ , is an injective homomorphism from the subgroup  $B_r := \{\alpha \in K : |\alpha| < r\}$  of  $K^+$  into  $\text{Aut}_{\text{cont}}(K_r/K)$ . Since  $p \nmid d$ ,  $\alpha \mapsto d\alpha$  is a bijection on  $B_r$ .

Now, if  $f^{\sigma_\alpha}(T) = \sum_{n \in \mathbb{Z}} b_n T^n$ , then

$$b_n = \sum_{m \geq n} \left( (-1)^{m-n} \binom{m}{n} + \binom{-n-1}{-m-1} \right) a_m \alpha^{m-n},$$

where we set  $\binom{a}{b} = 0$  if  $a < 0$  or  $b < 0$ . Suppose  $f^{\sigma_\alpha} = f$  for all  $\alpha \in B_r$ . Then in the above formula, we must have  $b_n = a_n$  for all  $\alpha \in B_r$ . This can only happen if  $a_n = 0$  for all  $n \neq 0$ . Therefore  $f \in K$ , and we may take  $H$  to be the image of  $B_r$  in  $\text{Aut}_{\text{cont}}(K_r/K)$ .  $\square$

**Lemma 2.5.** *Let  $X$  be a reduced affinoid over  $K$ , and let  $f : X \rightarrow A_K[a, b]$  be finite, flat, of degree  $d$ , where  $p \nmid d$  and  $a, b \in \mathcal{R}$  with  $a < 1 < b$ . Let  $T$  be the natural parameter on  $A_K[a, b]$ . Suppose there exists a function  $G$  on  $X[1] := f^{-1}C_K[1]$  such that  $\|G^d - f^*T\|_{X[1]} < 1$ . Then there exist  $a_1, b_1 \in \mathcal{R}[a, b]$  with  $a_1 < 1 < b_1$ , and a function  $S$  on  $f^{-1}A_K[a_1, b_1]$ , such that  $S^d = f^*T$ .*

*Proof.* Setting  $s = \bar{G}$  and  $t = \bar{T}$ , we have  $\mathcal{O}(\overline{X[1]}) = \mathbb{F}_K[s, s^{-1}]$ ,  $\mathcal{O}(\overline{C[1]}) = \mathbb{F}_K[t, t^{-1}]$ , and  $\bar{f} : \overline{X[1]} \rightarrow \overline{C[1]}$  given by  $t = s^d$ . Let  $V = A_K[a, 1]$  and  $U = f^{-1}(V)$ . Then  $\overline{C_K[1]}$  is an affine open in  $\overline{V}$ . Therefore, identifying  $\mathcal{O}(\overline{U})$  with its image in  $\mathcal{O}(\overline{X[1]})$ , we have

$$\mathcal{O}(\overline{X[1]}) = \mathcal{O}(\overline{U}) \otimes_{\mathcal{O}(\overline{V})} \mathcal{O}(\overline{C_K[1]}).$$

It follows that  $s$  is in the image of  $\mathcal{O}(\overline{U})$ . So we may lift  $s$  to a function  $S_0 \in A^\circ(U)$  such that

$$\|S_0^d - f^*T\|_{X[1]} < 1.$$

Now, choose  $a_1 \in \mathcal{R}[a, 1)$  such that  $|S_0^d - f^*T| < |f^*T|$  on  $U_1 := f^{-1}A_K[a_1, 1]$ . Let  $p(x) = x^d - (f^*T/S_0^d)$ , considered as a polynomial over  $A^\circ(U_1)$ . Then  $x_0 := 1$  satisfies  $|p(x_0)| < 1$  and  $|p'(x_0)| = 1$  over all of  $U_1$ . Therefore, by the usual Hensel's Lemma argument, there exists a unique  $x \in A^\circ(U_1)$  with  $p(x) = 0$  and  $\|x - 1\|_{U_1} < 1$ . Letting  $S_1 = S_0x$ , we have an  $S_1 \in A(U_1)$  whose restriction to  $X[1]$  is a lift of  $s$ , and for which  $S_1^d = f^*T$ .

By precisely the same argument, there exists a function  $S_2 \in A(U_2)$  which reduces to  $s$  on  $X[1]$  and satisfies  $S_2^d = f^*T$ , where  $U_2 = f^{-1}A_K[1, b_1]$  for some  $b_1 \in \mathcal{R}(1, b]$ . Moreover, since  $X$  is reduced,  $(S_1/S_2)^d = 1$  on  $X[1]$  (with  $p \nmid d$ ), and  $\|S_i - G\|_{X[1]} < 1$ , we must have  $S_1 = S_2$  on  $X[1]$ . Thus,  $S_1$  and  $S_2$  patch to a function  $S$  on  $f^{-1}A_K[a_1, b_1]$  with  $S^d = f^*T$ .  $\square$

**Theorem 2.6.** *Suppose  $a < b \in \mathcal{R}$ . Any finite connected étale cover over  $K$  of the annulus  $A_K[a, b]$  (resp.  $A_K(a, b)$ ) of degree  $d$ , where  $d < p$  if  $p \neq 0$ , is an annulus isomorphic over  $K$  to  $A_K[a^{1/d}c, b^{1/d}c]$  (resp.  $A_K(a^{1/d}c, b^{1/d}c)$ ) for some  $c \in |K^*|^{1/d}$ .*

*Proof.* We will first prove the statement for closed annuli.

Let  $W$  be a connected rigid space over  $K$ , and let  $f : W \rightarrow A_K[a, b]$  be finite, étale of degree  $d < p$  (if  $p \neq 0$ ). Initially, we also assume that  $a, b \in |K^*|$ . For each  $r \in |K^*| \cap [a, b]$ , let  $W_r$  be the inverse image in  $W$  of the circle  $C_K[r]$ . Then the connected components of  $W_r$ , which we denote by  $\{V_{r_1}, \dots, V_{r_{m_r}}\}$ , are affinoids over  $K$ , with each  $V_{r_i}$  finite and étale of degree  $d_{r_i}$  over  $C_K[r]$ , such that  $\sum d_{r_i} = d$ . As  $d < p$  or  $p = 0$ , each  $V_{r_i}$  must be finite and étale of degree  $d_{r_i}$  over  $C_K[r] \cong \mathbf{G}_m$ . Thus, there must exist an isomorphism,  $\sigma_{r_i} : V_{r_i} \rightarrow C_K[r^{1/d_{r_i}}]$ , such that  $f \circ \sigma_{r_i}^{-1}$  reduces to  $x \mapsto x^{d_{r_i}}$  on  $\mathbf{G}_m$  (with respect to the standard parameters). Moreover, this implies by Lemma 2.5 that for each  $r \in |K^*| \cap (a, b)$  there exist  $\alpha_r, \beta_r \in \mathcal{R}[a, b]$  with  $\alpha_r < r < \beta_r$ , and an embedding

$$F_r : \prod_{i=1}^{m_r} A_K(\alpha_r^{1/d_{r_i}}, \beta_r^{1/d_{r_i}}) \hookrightarrow W,$$

such that  $\text{Im}(F_r) = f^{-1}A_K(\alpha_r, \beta_r)$ . In fact,  $F_r^{-1}$  can be defined on the  $i^{\text{th}}$  component of  $f^{-1}A_K(\alpha_r, \beta_r)$  by a parameter  $S_{r_i}$  such that  $S_{r_i}^{d_{r_i}} = f^*T$  (where  $T$  is the natural parameter on  $A_K(\alpha_r, \beta_r)$ ). Similarly, we have embeddings,  $F_a$  and  $F_b$ , each of a disjoint union of semi-open annuli into  $W$ , with images  $f^{-1}A_K[a, \beta_a)$  and  $f^{-1}A_K(\alpha_b, b]$ .

Now, suppose further that  $[a, b] = [a, \beta_a) \cup (\alpha_b, b] \cup \bigcup_{r \in |K^*| \cap (a, b)} (\alpha_r, \beta_r)$ . Then by compactness of  $[a, b]$ , we may choose a finite set,  $\{r_1, \dots, r_n\} \subset |K^*| \cap (a, b)$  such that  $[a, b]$  is covered by  $[a, \beta_a)$ ,  $(\alpha_b, b]$  and the intervals,  $(\alpha_{r_i}, \beta_{r_i})$ , for  $1 \leq i \leq n$ . Whenever two of these intervals overlap, it is clear from the properties of  $F_r$  that the inverse images in  $W$  of the corresponding subannuli of  $A_K[a, b]$  must have the same number of connected components. Therefore, as  $W$  is connected, it follows that  $m_r = 1$  for all  $r \in |K^*| \cap [a, b]$ . Thus,  $F_r$  is an isomorphism of  $A_K(\alpha_r^{1/d}, \beta_r^{1/d})$  onto  $f^{-1}A_K(\alpha_r, \beta_r)$ , given by a parameter  $S_r$  with  $S_r^d = f^*T$  (for  $r \in |K^*| \cap (a, b)$ , and similarly for  $r = a$  or  $b$ ). We claim that  $F_a$ ,  $F_b$ , and the  $F_{r_i}$  can be used to construct an isomorphism of  $A_K[a^{1/d}, b^{1/d}]$  onto  $W$ . Indeed, whenever  $(\alpha_{r_i}, \beta_{r_i}) \cap (\alpha_{r_j}, \beta_{r_j}) = (\alpha_{r_j}, \beta_{r_i})$ , we have a parameter  $S_{r_i}$  on  $f^{-1}A_K(\alpha_{r_i}, \beta_{r_i})$  such that  $S_{r_i}^d = f^*T$ , and likewise for  $r_j$ . After adjusting by a  $d^{\text{th}}$  root of unity in  $K$  if necessary,  $S_{r_i}$  and  $S_{r_j}$  agree on the annulus,  $f^{-1}A_K(\alpha_{r_j}, \beta_{r_i})$ . Therefore the two parameters patch to a parameter  $S_{i_j}$  which identifies  $f^{-1}A_K(\alpha_{r_i}, \beta_{r_j})$  with  $A_K(\alpha_{r_i}^{1/d}, \beta_{r_j}^{1/d})$ . Moreover, after finitely many such patching steps, we have constructed a parameter  $S$  on  $W$  over  $K$  such that  $S^d = f^*T$  and thus defines an isomorphism from  $W$  onto  $A_K[a^{1/d}, b^{1/d}]$ .

More generally, without making the above two suppositions, for each  $r \in [a, b]$  take  $M_r$  to be a finite Galois extension of  $K$  such that  $r \in |M_r^*|$  if  $r \in \mathcal{R}$  and  $K_r$  (defined as above) otherwise. Then we may choose  $\alpha_r, \beta_r \in \mathcal{R}_{M_r}[a, b]$  and an embedding  $F_r$  which is defined over  $M_r$ , precisely as was done above over  $K$ . Now, we *know* that  $[a, b]$  is covered by  $[a, \beta_a)$ ,  $(\alpha_b, b]$ , and  $\{(\alpha_r, \beta_r) : r \in (a, b)\}$ . So by compactness, there exists a finite set  $t_1, \dots, t_m \in (a, b)$  such that  $[a, b]$  is covered by  $[a, \beta_a)$ ,  $(\alpha_b, b]$ , and  $\{(\alpha_{t_i}, \beta_{t_i}) : 1 \leq i \leq m\}$ . Choose a finite Galois

extension  $L$  of  $K$  so that the images of the  $t_i$  in  $\mathbb{R}_+^*/|L^*|$  generate a torsion free Abelian group. Then choose  $r_1, \dots, r_n \in \mathbb{R}_+^*$  so that their images form a basis for this group. Then the above argument can be applied to produce a parameter  $S$  on  $W$ , which is defined over  $L_{r_1, \dots, r_n}$ , such that  $S^d = f^*T$ .

Now, if  $\sigma \in \text{Aut}_{\text{cont}}(L_{r_1, \dots, r_n}/L)$  the map  $\sigma \mapsto \zeta(\sigma) := S^\sigma/S$  is a one-cocycle with values in  $\mu_d(A(W_{L_{r_1, \dots, r_n}}))$ . Since  $W$  is connected this equals  $\mu_d(L_{r_1, \dots, r_n})$ , which is  $\mu_d(L)$ . It follows from Lemma 2.4 that  $\zeta(\sigma) = 1$  for all  $\sigma$  in a subgroup whose fixed field is  $L$ . Thus  $S$  is defined over  $L$ . Then, for  $\sigma \in \text{Gal}(L/K)$ ,  $S^\sigma = h(\sigma)S$ , where  $h$  is a one-cocycle. So by Hilbert's Theorem 90 there exists  $\gamma \in L^*$  such that  $h(\sigma) = \gamma^\sigma/\gamma$ . Then  $H := S/\gamma$  is defined over  $K$  and  $H^d = \alpha T$  for some  $\alpha \in K^*$ . Therefore  $H$  defines an isomorphism of  $W$  onto  $A_K[a^{1/d}c, b^{1/d}c]$ , where  $c = |\alpha|^{1/d}$ .

To deal with open annuli  $A_K(a, b)$  choose sequences  $\{a_n\}$  and  $\{b_n\}$  in  $\mathcal{R}_K(a, b)$  such that  $a_n < b_n$ ,  $a_n \rightarrow a$  and  $b_n \rightarrow b$ . For large  $n$ ,  $W_{[a_n, b_n]} := f^{-1}A_K[a_n, b_n]$  is connected, and it is finite, étale over  $A_K[a_n, b_n]$  of degree  $d$ . Therefore, it is isomorphic to  $A_K[a_n^{1/d}c_n, b_n^{1/d}c_n]$  by what we have proven. The theorem follows when we let  $n$  go to infinity.  $\square$

**Remark 2.7.** (i) When  $K$  is algebraically closed, Lemma 2.3 of [L] implies that there exists  $a = c_0 < \dots < c_{n+1} = b$  in  $\mathcal{R}$  such that  $f^{-1}A(c_i, c_{i+1})$  is a disjoint union of open annuli. One could then use Hilbert's Theorem 90 and Lemma 2.5, as in the above proof, to give another proof of the theorem.

(ii) One can obtain the same conclusion about  $W$ , for any finite étale surjection  $f$  whose Galois closure has degree prime to  $p$  when  $p \neq 0$ .

If  $X$  is a reduced affinoid over  $K$  and  $P \in \overline{X}(\mathbb{F}_K)$ , we let  $R_X(P)$  denote the **residue class** of  $P$ . When the context makes it clear, we will drop the subscript  $X$ . This is the open rigid subspace of  $X$  whose  $\mathbf{C}$  valued points reduce to  $P$ , equivalently the subspace  $\text{Red}^{-1}P$  where  $P$  is naturally identified with a subscheme of  $\overline{X}$ . Alternatively, suppose  $f_1, \dots, f_m \in A^\circ(X)$  are such that  $\bar{f}_1, \dots, \bar{f}_m$  generate the maximal ideal of  $P$ . Then  $R(P)$  is admissibly covered by the increasing sequence of affinoids whose  $\mathbf{C}$  valued points are

$$\{x \in X(\mathbf{C}) : |f_i(x)| \leq r_n, 1 \leq i \leq m\},$$

where  $r_n \in \mathcal{R}$ ,  $r_n < r_{n+1}$  and  $\lim_{n \rightarrow \infty} r_n = 1$ . If  $x$  is a point of  $X$  such that  $\bar{x} = P$  (which always exists by Theorem 6.4 of [T2]), this is naturally isomorphic to the formal fiber  $X_+(x)$  studied in [Bo3], by Satz 6.1 of [Bo3].

**Proposition 2.8.** *Let  $K$  be a stable field. Suppose  $X$  is a reduced pure  $d$ -dimensional affinoid over  $K$ ,  $||A(X)|| = |K|$  (equivalently,  $A^\circ(X) \otimes_{R_K} \mathbb{F}_K$  is reduced) and  $P \in \overline{X}(\mathbb{F}_K)$ . Then  $\overline{A(R(P))} \cong \hat{\mathcal{O}}_{\overline{X}, P}$ .*

*Proof.* Let  $I(P)$  be the closure of  $m_K A^\circ(R(P))$  in  $A^\circ(R(P))$ . On page 44 of [Bo3], Bosch proves that  $A^\circ(R(P))/I(P) \cong \hat{\mathcal{O}}_{\overline{X}, P}$  when there exists a surjective map  $\phi : T_n \rightarrow A(X)$  such that  $\hat{\phi}$  is surjective.<sup>2</sup> That such a map exists when

<sup>2</sup>As the example at the end of [BGR, §6.4] implies,  $\phi$  need not be distinguished (see [BGR, Def. 6.4.3/2]).

$K$  is stable and  $\|A(X)\|_X = |K|$ , follows from Corollary 6.4.3/6 of [BGR]. It is clear that  $I(P) \subseteq A^+(R(P)) \subseteq \text{rad}(I(P))$ . Since  $\bar{X}$  is reduced, so is  $\hat{\mathcal{O}}_{\bar{X},P}$ , and hence  $I(P) = A^+(R(P))$ . The proposition follows.  $\square$

**Definition 2.9.** *Let  $P$  be a point on a curve  $C$  over a field  $k$ . We say that  $P$  is an ordinary double point over  $k$  if  $\hat{\mathcal{O}}_{C,P} \cong k[[u, v]]/(uv)$ .*

We say that  $K$  satisfies **Hypothesis B** if  $R_K$  contains a bald subring (see [BGR, Def. 1.7.2/1]) with the same residue field. This is the case if  $K$  is discretely valued, if its residue field is perfect, or if its residue field lifts to a subfield. In particular, this is the case if  $K$  satisfies Hypothesis T. We do not know if all complete, non-archimedean valued fields  $K$  satisfy Hypothesis B.

**Proposition 2.10.** *Let  $X$  be a reduced, irreducible affinoid over a stable field  $K$  satisfying Hypothesis B. Suppose that  $\bar{X}$  is a reduced curve and  $P \in \bar{X}(\mathbb{F}_K)$ . Then  $P$  is an ordinary double point over  $\mathbb{F}_K$  if and only if the residue class,  $R(P)$ , is isomorphic to  $A_K(r, 1)$  for some  $r \in |K^*|$ .*

This was proven in [BL1, Prop. 2.3] when  $K$  is algebraically closed, and we adapt their proof to our case here.

**Lemma 2.11.** *Suppose  $I$  is a bald subring of  $R_K$  and  $\{r_1, r_2, \dots\}$  is a zero sequence in  $R_K$ . Then there exists a bald subring of  $R_K$  containing  $I$  and  $r_n$  for all  $n \geq 1$ .*

*Proof of Lemma 2.11.* The proof is almost identical to that of Corollary 1.7.2/5 of [BGR] (just replace the  $I$  in the proof of 1.7.2/4 with this  $I$ ).  $\square$

**Lemma 2.12.** *Let  $X$  be a reduced, one dimensional affinoid, with reduced reduction, over a stable field  $K$  satisfying Hypothesis B. Suppose that  $f, g \in C := A(\bar{X})$  generate a maximal ideal  $\mathcal{M} = (f, g)$ , such that  $C/\mathcal{M} \cong \mathbb{F}_K$  and  $fg \in \mathcal{M}^3$ . Let*

$$U = \begin{cases} X & \text{if } fg = 0 \\ \{x \in X : f(\bar{x}) = g(\bar{x}) = 0\} & \text{otherwise.} \end{cases}$$

*Then there exist  $F, G \in A^\circ(U)$  and  $c \in R_K$ , such that  $\bar{F} - f \in \mathcal{M}^2 \overline{A(U)}$ ,  $\bar{G} - g \in \mathcal{M}^2 \overline{A(U)}$  (using Proposition 2.8 to identify  $\overline{A(U)}$  with  $\hat{\mathcal{O}}_{\bar{X},U}$ ), and  $FG = c$ .*

*Proof of Lemma 2.12.* Suppose that  $f_1, g_1 \in A^\circ(X)$  are such that  $f = \bar{f}_1$  and  $g = \bar{g}_1$ , and that  $\alpha : X \rightarrow B_K[1]$  is a finite morphism. That  $\bar{X}$  is reduced implies  $\|A(X)\|_X = |K|$ . So by Corollary 6.4.1/4 of [BGR],  $\alpha^* : A^\circ(B_K[1]) \rightarrow A^\circ(X)$  is finite. Now suppose  $\alpha^*(A^\circ(B_K[1])) = R_K\langle T \rangle$ . As  $C$  is torsion free (because  $\bar{X}$  is flat over  $\mathbf{A}^1$ ) and finitely generated over  $\mathbb{F}_K[T]$ , it is free. Choose  $h_1, \dots, h_n \in A^\circ(X)$  so that  $\bar{h}_1, \dots, \bar{h}_n$  is a basis for  $C$  over  $\mathbb{F}_K[T]$ . Then  $h_1, \dots, h_n$  is a basis for  $A^\circ(X)$  over  $R_K\langle T \rangle$ . It follows that  $B := \{h_i T^j : 1 \leq i \leq n, j \in \mathbb{N}_0\}$  is

an orthonormal Schauder basis [BGR, Def. 2.7.2/1 and page 117] for  $A(X)$ . As  $C = \mathcal{M} \oplus_{\mathbb{F}_K} \mathbb{F}_K$ , the ring  $C$  has a basis over  $\mathbb{F}_K$  of the form  $\{1, \bar{\alpha}_i f, \bar{\beta}_j g : i, j \in \mathbb{N}\}$  for some  $\alpha_i, \beta_j$  in a subring of  $A^\circ(X)$  finitely generated over a bald subring of  $R_K$  with residue field  $\mathbb{F}_K$ . It follows from Lemma 2.11 that the change of basis matrix from  $B$  to  $\{1, \alpha_i f_1, \beta_j g_1 : i, j \in \mathbb{N}\}$  has entries in a bald subring of  $R_K$  [BGR, Def. 1.7.2/1]. Hence by the Lifting Theorem of [BGR, Thm. 2.7.3/2] this is also an orthonormal Schauder basis. Hence  $A^\circ(X) = R_K + M$  where  $M = (f_1, f_2)$ .

We have

$$f_1 g_1 = \pi c_1 + f_1(\pi a_1 + b_1) + g_1(\pi a_2 + b_2)$$

with  $c_1 \in R_K$ ,  $a_i \in A^\circ(X)$ ,  $b_i \in M^2$  (and  $b_i = 0$  if  $fg = 0$ ) for some  $\pi \in R_K$ ,  $|\pi| < 1$ . Let  $I = \pi R_K + f_1 A^\circ(X) + g_1 A^\circ(X) = \pi A^\circ(X) + M$ , and let  $J = \pi A^\circ(X)$  if  $fg = 0$  and  $I$  otherwise. Let  $f_2 = f_1 - (\pi a_2 + b_2)$ , and  $g_2 = g_1 - (\pi a_1 + b_1)$ . Then

$$\begin{aligned} f_2 g_2 &= \pi c_1 + (\pi a_1 + b_1)(\pi a_2 + b_2) \\ &\equiv \pi c_1 + \pi^2 c'_2 \pmod{(\pi A^\circ(X) + M)^3} \\ &\equiv \pi c_1 + \pi^2 c'_2 \pmod{\pi^2 M} \quad \text{if } fg = 0 \end{aligned}$$

for some  $c'_2 \in R_K$ . Now  $I^n = \pi^n R_K + f_1 I^{n-1} + g_1 I^{n-1}$ , so this implies

$$f_2 g_2 = \pi c_1 + \pi^2 c_2 + f_1 r_{2,1} + g_1 r_{2,2}$$

where  $c_2 \in R_K$ ,  $r_{2,i} \in J^2$ . Let  $k_n = 2^{n-2} + 1$ , for  $n \geq 2$  and  $k_1 = 1$ . Suppose

$$f_n g_n = \pi c_1 + \pi^2 c_2 + \pi^4 c_3 + \cdots + \pi^{2k_{n-1}} c_n + f_1 r_{n,1} + g_1 r_{n,2}$$

for some  $r_{n,i} \in J^{k_n}$ . Set  $f_{n+1} = f_n - r_{n,2}$  and  $g_{n+1} = g_n - r_{n,1}$ . Then

$$\begin{aligned} f_{n+1} g_{n+1} &= \pi c_1 + \pi^2 c_2 + \pi^4 c_3 + \cdots + \pi^{2k_{n-1}} c_n + r_{n,1} r_{n,2} \\ &= \pi c_1 + \pi^2 c_2 + \pi^4 c_3 + \cdots + \pi^{2k_{n-1}} c_n + \pi^{2k_n} c_{n+1} + f_1 r_{n+1,1} + g_1 r_{n+1,2} \end{aligned}$$

where  $r_{n+1,i} \in J^{k_{n+1}}$ .

Finally, let  $r_{1,1} = \pi a_1 + b_1$  and  $r_{1,2} = \pi a_2 + b_2$ . Set  $F = f_1 - \sum_{n \geq 1} r_{n,2}$  and  $G = g_1 - \sum_{n \geq 1} r_{n,1}$ . Then these are elements of  $A^\circ(U)$  which satisfy

$$FG = c := \pi c_1 + \pi^2 c_2 + \sum_{n \geq 3} \pi^{2^{n-2}+2} c_n.$$

□

*Proof of Proposition 2.10.* Suppose  $P \in \bar{X}(\mathbb{F}_K)$  is an ordinary double point. We can apply the above lemma to conclude that there exist  $F, G \in A^\circ(R(P))$  and  $c \in R_K$ , such that  $(\bar{F}, \bar{G}) = \mathcal{M}_P$  and  $FG = c$ . Thus we have a morphism

$$R(P) \rightarrow A_K(|c|, 1).$$

That this is an isomorphism follows, as in the proof of [BL1, Prop. 2.3].

Conversely, suppose that  $R(P)$  is isomorphic to the annulus,  $A_K(r, 1)$ , for some  $r \in |K^*|$  with  $r < 1$ . Then  $A^\circ(R(P)) \cong R_K[[T, cT^{-1}]]$ , where  $c \in K$  with  $|c| = r$ . So applying Proposition 2.8 we have

$$\hat{\mathcal{O}}_{\bar{X}, P} \cong \overline{A(R(P))} \cong \mathbb{F}_K[[x, y]]/(xy),$$

and hence  $P$  is an ordinary double point of  $\bar{X}$ .  $\square$

For a rigid space  $W$  over  $K$  set

$$D^i(W/K) = \text{Ker}(d: \Omega_{W/K}^i(W) \rightarrow \Omega_{W/K}^{i+1}(W))/d\Omega_{W/K}^{i-1}(W),$$

where if  $A(W)$  is the ring of rigid functions on  $W$ ,  $\Omega_{W/K}^i(W)$  is the  $A(W)$  module of rigid  $i$ -forms on  $W$ .

**Lemma 2.13.** *Suppose  $W = A_K(r, s)$  or  $B_K(r) \setminus \{0\}$ , where  $r, s \in |L^*|$  for some finite extension  $L/K$ . Then*

$$D^0(W/K) \cong D^1(W/K) \cong K.$$

*Proof.* If  $r, s \in |K^*|$ , the lemma is clear. For in this case, we may choose  $a, b \in K$  with  $|a| = r$  and  $|b| = s$ , and let  $x = T/b$  and  $y = a/T$  where  $T$  is the natural parameter on  $\mathbf{A}_K^1$ . Then  $A_K(W)$  is equal to the set of functions represented by

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} b_n y^n,$$

where  $a_n, b_n \in K$ ,  $a_n t^n \rightarrow 0$  and  $b_n t^n \rightarrow 0$  as  $n \rightarrow \infty$ , for  $|t| < 1$ . Moreover, there is a natural continuous linear map  $\rho_K$  from  $\Omega_{W/K}^1(W)$  onto  $K$  such that  $\rho_K(dv/v) = 1$  for any parameter  $v$  on  $W_K$  such that  $|v(u)| > |v(w)|$  if  $|u| > |w|$ ,  $u, w \in A_K(r, s)(\mathbf{C})$ . Moreover, for any  $\omega \in \Omega_{W/K}^1(W)$ ,  $\omega \in dA_K(W)$  if and only if  $\rho_K(\omega) = 0$ .

More generally, suppose  $L$  is a finite Galois extension of  $K$  with Galois group  $G$ , and that  $r, s \in |L^*|$ . Then  $G$  acts on  $\Omega^1(A_L(r, s))$ , such that  $\Omega^1(A_L(r, s))^G = \Omega^1(A_K(r, s))$  and

$$\rho_L(\omega^\sigma) = \rho_L(\omega)^\sigma.$$

Also, if  $r, s \in |K^*|$ , then  $\rho_L|_{\Omega^1(A_K)} = \rho_K$ . Suppose  $\omega \in \Omega^1(A_K(r, s))$  and  $\rho_L(\omega) = 0$ . Then using the additive Hilbert's Theorem 90,  $\omega \in dA(A_K(r, s))$ . Thus we have an injective  $K$ -linear map  $D^1(W/K) \rightarrow L^G = K$ . If  $\omega \in \Omega^1(A_L(r, s))$ ,  $\rho_L(\omega) = 1$  and  $\nu = \sum_{\sigma \in G} \omega^\sigma$ , then  $\nu \in \Omega^1(A_K(r, s))$  and  $\rho_L \nu = [L : K]$ . So this map is an isomorphism.  $\square$

From the proof we see that for any open annulus  $W$  over  $K$  there are two "residue" maps from  $\Omega_{W/K}^1$  onto  $K$ . In particular, they are  $\text{res}_{r,s} \circ f^*$  and  $-\text{res}_{r,s} \circ f^*$ , where  $f: A_L(r, s) \rightarrow W_L$  is an isomorphism and  $\text{res}_{r,s} = \rho_L|_{\Omega^1(A_K)}$  for any

extension  $L$  of  $K$  such that  $r, s \in |L^*|$ . By an oriented annulus over  $K$  we mean a pair  $(W, \rho)$ , where  $W$  is an open annulus and  $\rho$  is a choice of one of the residue maps.

An **end** of a rigid space  $W$  over  $K$  is an element of the inverse limit of the set of connected components of  $W \setminus Z$ , where  $Z$  ranges over finite unions of subaffinoids of  $W$  defined over  $K$  (ordered by containment). We let  $\mathcal{E}(W)$  denote the set of ends of  $W$  and  $e(W) = |\mathcal{E}(W)|$  (which may be infinite). For example,  $e(W) = 2$  whenever  $W$  is an open annulus. If  $W$  is admissibly covered by a countable number of affinoids, and  $f$  is a real-valued function of  $W(\mathbf{C})$ , it makes sense to compute the limit of  $f$  at an end,  $e \in \mathcal{E}(W)$ . In particular, we define  $\lim_{x \rightarrow e} f(x) = \lim_{n \rightarrow \infty} f(x_n)$ , where  $\{x_n\}$  is any sequence in  $W(\mathbf{C})$  such that for any  $Z$  as above,  $x_n$  is contained in a connected component of  $W \setminus Z$  which maps to  $e$  for sufficiently large  $n$  (provided this limit exists and is independent of the sequence).

The following result is used in the proof of [CMc, Theorem 5.2].

**Proposition 2.14.** *Suppose  $K$  is discretely valued and  $U$  is a rigid space over  $K$  with two ends such that for some finite extension  $L$  of  $K$ ,  $U_L$  is isomorphic to the open annulus  $A_L(|u|, 1)$ , where  $u \in K^*$ ,  $|u| < 1$ . Then  $U \cong A_K(|u|, 1)$ .*

*Proof.* We may suppose that  $e(L/K) > 1$  and  $L$  is a Galois extension of  $K$  with Galois group  $G$ . Let  $M = A(U_L)$ . Then  $G$  naturally acts on  $M$ , and  $M^G = A(U)$ . Let  $R = R_L$ , and let  $\mathbb{F}_L$  and  $\mathbb{F}_K$  denote the residue fields of  $L$  and  $K$ . Let  $\pi$  be a uniformizing parameter on  $L$ .

Let  $a$  and  $b$  denote the ends of  $U$ , and suppose  $F \in M$  is an isomorphism from  $U_L$  onto  $A_L(|u|, 1)$  such that  $\lim_{x \rightarrow a} |F(x)| = 1$ . Then we may use  $F$  to identify  $M$  with  $L\{\{T, u/T\}\}$  and the group of orientation-preserving automorphisms of  $A_L(|u|, 1)$ , denoted  $M_a$ , with

$$\{T(\sum_{i=0}^{\infty} a_i T^i + \sum_{i=1}^{\infty} b_i (u/T)^i) : a_i, b_i \in R \text{ and } a_0 \in R^*\}$$

(under composition). Note that the group  $G$  preserves  $M_a$ . For  $\sigma \in G$ , set  $\sigma(T) = G_\sigma(T)$ . For  $h(T) = \sum_{i=0}^{\infty} a_i T^i + \sum_{i=1}^{\infty} b_i (u/T)^i \in L\{\{T, u/T\}\}$ , set  $h^\sigma(T) = \sum_{i=0}^{\infty} a_i^\sigma T^i + \sum_{i=1}^{\infty} b_i^\sigma (u/T)^i$ . Then

$$G_\tau^\sigma \circ G_\sigma = G_{\sigma\tau}. \quad (2.0)$$

We will show that there exists  $F \in M_a$  such that

$$F^\sigma \circ F^{-1} = G_\sigma. \quad (2.1)$$

This will imply that  $F^{-1}(T) \in A(U)$ , and as  $F^{-1}(T)$  is a parameter on  $U$ , it will then follow that  $U$  is an annulus over  $K$ .

So first let  $I$  be the ideal in  $C := R[[T, u/T]]$  generated by  $\pi$ ,  $T$  and  $u/T$ , and suppose that

$$G_\sigma(T) \equiv a(\sigma)T \pmod{TI}$$

where  $a(\sigma) \in R^*$ . Then, from (2.0), we have

$$a(\sigma)^\tau a(\tau) \equiv a(\sigma\tau) \pmod{\pi}.$$

Using Hilbert's Theorem 90 applied to  $\mathbb{F}_L/\mathbb{F}_K$ , one can show there exists a  $c \in R^*$  such that

$$c^\sigma/c \equiv a(\sigma) \pmod{\pi}.$$

Let  $h(T) = cT$ . Then we have

$$(h^{-\sigma} \circ G_\sigma \circ h)(T) \equiv T \pmod{TI}.$$

Now, suppose  $G_\sigma(T) = T(1 + h_\sigma(T))$ , where  $h_\sigma(T) \in I^k$ .

**Lemma 2.15.** *Suppose  $h(T) := \sum_{i=1}^{\infty} B_{-i}(u/T)^i + \sum_{i=0}^{\infty} B_i T^i$  is in  $C$ . Then  $h(T) \in I^k$  if and only if  $B_i \equiv 0 \pmod{\pi^{k-|i|}R}$ .*

*Proof of Lemma 2.15.* Let  $S_k$  be the  $R$  module of series whose coefficients satisfy the above bounds. The lemma is clearly true for  $k = 0$ . Suppose it is true for  $k$ . Let  $\mathcal{T}$  be the continuous involution of the  $R$ -algebra  $C$  which takes  $T$  to  $u/T$ . Then  $I$  and  $S_k$  are preserved by  $\mathcal{T}$ . As  $\pi^{k-i}T^i \in I^k$ ,  $0 \leq i \leq k$ , and  $T^{k+1}R[[T]] \subseteq I^{k+1}$ , it follows that  $S_{k+1} \subseteq I^{k+1}$ . Now, we have

$$\pi(\pi^{k-i}T^i) = \pi^{k+1-i}T^i, \quad T(\pi^{k-i}T^i) = \pi^{k+1-(i+1)}T^{i+1},$$

and because  $v(u) \geq 2v(\pi)$ ,

$$T(\pi^{k-i}(u/T)^i) = u\pi^{k-i}(u/T)^{i-1} \in \begin{cases} \pi^{k+1-(i-1)}(u/T)^{i-1}R & \text{if } i > 0 \\ \pi^{k+1-1}TR & \text{if } i = 0. \end{cases}$$

Thus  $I^{k+1} \subseteq S_{k+1}$ . □

Now suppose

$$h_\sigma(T) = \sum_{i=1}^{\infty} B_{-i}(\sigma)(u/T)^i + \sum_{i=0}^{\infty} B_i(\sigma)T^i.$$

Then, since

$$T(1 + h_\tau(T))(1 + h_\sigma^\tau(T(1 + h_\tau(T)))) \equiv T(1 + h_\sigma^\tau(T) + h_\tau(T)) \pmod{TI^{2k}},$$

it follows that

$$\begin{aligned} G_\sigma^\tau \circ G_\tau(T) &\equiv T\left(1 + \sum_{i=1}^{2k} (B_{-i}(\tau) + B_{-i}(\sigma)^\tau)(u/T)^i\right. \\ &\quad \left. + \sum_{i=0}^{2k} (B_i(\tau) + B_i(\sigma)^\tau)T^i\right) \pmod{TI^{2k}}. \end{aligned}$$

Therefore, by Lemma 2.15 we have

$$B_i(\sigma\tau) \equiv B_i(\tau) + B_i(\sigma)^\tau \pmod{\pi^{2k-|i|}}.$$

Finally, using Hilbert's Theorem 90 again, we can find  $C_i \in \pi^{k-|i|}R \cap R$  such that

$$C_i^\tau - C_i \equiv B_i(\tau) \pmod{\pi^{2k-|i|}},$$

for  $-2k \leq i \leq 2k$ . So let

$$H(T) = T \left( 1 + \sum_{i=1}^{2k} C_{-i} (u/T)^i + \sum_{i=0}^{k^2-1} C_i T^i \right).$$

Then  $H \in TI^k$  and  $H^\sigma \circ H^{-1} \equiv G_\sigma \pmod{TI^{2k}}$ . Thus we can find a convergent sequence  $F_k \in M_a$  such that  $F_k^\sigma \circ F_k^{-1} \rightarrow G_\sigma$  in  $M_a$ . The limit,  $F \in M_a$ , must satisfy (2.1).  $\square$

**Remark 2.16.** *Suppose  $K$  is discretely valued and  $U$  is a rigid space with one end over  $K$ , such that  $U_L$  is isomorphic to the open disk  $B_L(1)$  for some finite extension  $L$  of  $K$ . Then it follows from a similar argument that  $U \cong B_K(1)$ .*

## 2.2 Wide Open Spaces

In [C1, §III] we defined wide open spaces over  $\mathbb{C}_p$ . Now we need to use them over more general fields. Suppose  $W$  is a one dimensional smooth rigid space over  $K$ . Then  $W$  is a wide open space, or wide open, over  $K$ , if it contains affinoid subdomains  $X$  and  $Y$  such that:

- (i)  $W \setminus X$  is a disjoint union of finitely many open annuli
- (ii)  $X$  is relatively compact in  $Y$
- (iii)  $Y \cap V$  is a semi-open annulus for all connected components  $V$  of  $W \setminus X$ .

We call  $X$  an underlying affinoid of  $W$ . From the definition, it is immediate that there is a natural bijection between  $\mathcal{E}(W)$ , the set of ends of  $W$ , and  $CC(W \setminus X)$ , the set of connected components of  $W \setminus X$ . And  $X$  is connected to each element of  $CC(W \setminus X)$ . So  $e(W)$  is finite in this case. We call the connected component of  $W \setminus X$  which corresponds to an element  $e$  of  $\mathcal{E}(W)$  an **annulus at  $e$** .

**Remark 2.17.** *It is not immediate that the intrinsic definition of wide open space given above is equivalent to the one given in [C1, §III] when  $K = \mathbb{C}_p$ . However, this will follow in one direction from Theorem 2.18 and in the other from Theorem 2.40.*

**Theorem 2.18.** *Let  $W$  be a wide open over  $K$  with underlying affinoid  $X$ . Then  $W$  may be completed to a proper algebraic curve  $C$  over  $K$  by gluing open disks onto the connected components of  $W \setminus X$ .*

*Proof.* More specifically, let  $\mathcal{S}$  be the set of connected components of  $W \setminus X$ . For each open annulus  $V \in \mathcal{S}$ , let  $\alpha_V : V \rightarrow B_V$  be an embedding of  $V$  into an

open disk over  $K$  such that  $B_V \setminus \alpha_V(V')$  is connected for any concentric annulus  $V' \subseteq V$  which is connected to  $X$ . We will show that

$$C := (W \cup \prod_{V \in \mathcal{S}} B_V) / \{\alpha_V(V) = V\}_{V \in \mathcal{S}}$$

is isomorphic to a complete algebraic curve.

It is clear that  $C$  is smooth of dimension one. Therefore, to establish the claim, by the Riemann Existence Theorem (Theorem A.2), we need only show that  $C$  is proper [BGR, Def. 9.6.2/2]. The number of connected components of  $W$  is finite and equals the number of connected components of  $X$ , and so we may assume without loss of generality that  $W$  is connected. In this case  $X$  is contained in a residue class  $R(P)$  of  $Y$  (where  $P$  is the image of  $\bar{X}$  in  $\bar{Y}$ ). Choose an  $f \in A^\circ(Y)$  such that  $P$  is an isolated zero of  $\bar{f}$ . This can be done by first passing to a finite extension  $L$  of  $K$  so that  $\bar{Y}_L$  is reduced and so that there is such a function  $g \in A^\circ(Y_L)$ . Then let  $f$  be the norm of  $g$ . Now by Lemma 2.4 of [BL1], if  $\alpha \in \mathcal{R}$ ,  $\alpha < 1$  and sufficiently close to 1,  $\{x \in R(P) : |f(x)| \geq \alpha\}$  is the set of  $\mathbf{C}$ -valued points in a subdomain  $U_\alpha$  of  $W$  which, after a finite extension (the field in *ibid* is algebraically closed), becomes isomorphic to a finite union of semi-open annuli. In fact, for  $\alpha$  sufficiently close to 1,  $U_\alpha$  must decompose as  $\prod_{V \in \mathcal{S}} A_{\alpha,V}$ , where each  $A_{\alpha,V}$  is a concentric semi-open annulus in  $V$ . It follows that  $B_{1,V} := B_V \setminus \alpha_V(V \cap Y)$  and  $B_{\alpha,V} := B_{1,V} \cup \alpha_V(A_{\alpha,V})$  are closed disks. Also, we may define  $X_\alpha$  to be the rigid subdomain of  $W$  whose  $\mathbf{C}$ -valued points are  $\{x \in R(P) : |f(x)| \leq \alpha\}$ .

Then, for any  $\beta \in \mathcal{R}$  with  $\alpha < \beta < 1$ ,  $\mathcal{U} := \{X_\beta, B_{\beta,V} : V \in \mathcal{S}\}$  and  $\mathcal{V} := \{Y, B_{\alpha,V} : V \in \mathcal{S}\}$  are two finite admissible affinoid coverings of  $C$  such that each element of  $\mathcal{U}$  is relatively compact in an element of  $\mathcal{V}$ . So  $C$  is proper, if it is separated. To verify that  $C$  is separated, we must show that the diagonal map,  $\Delta : C \rightarrow C \times_K C$ , is a closed immersion. This can be checked locally using the admissible affinoid cover of  $C \times_K C$  given by  $\{Z \times_K Z' : Z, Z' \in \mathcal{U}\}$ . Indeed, for every  $Z, Z' \in \mathcal{U}$ ,  $\Delta^{-1}(Z \times_K Z') = Z \cap Z'$  is an affinoid and

$$\Delta^* : A(Z \times_K Z') \rightarrow A(Z \cap Z')$$

is surjective. This is obvious when  $Z = Z'$ . Otherwise, when  $Z \cap Z' \neq \emptyset$  we must have  $\{Z, Z'\} = \{X_\beta, B_{\beta,V}\}$  for some  $V \in \mathcal{S}$ . So in this case,  $Z \cap Z'$  is a circle over  $K$ , in particular the concentric circle in  $V \cap Y$  defined by  $|f(x)| = \beta$ . To obtain surjectivity, first make a finite base extension  $L$  of  $K$  so that  $(X_\beta)_L$  and  $(B_{\beta,V})_L$  are reduced. Then  $\mathcal{O}(\overline{(X_\beta)_L})$  is isomorphic to a subring of

$$\mathbb{F}_L[t_1, \dots, t_N] / (t_i t_j)_{i \neq j}$$

which contains a power of the ideal  $(t_1, \dots, t_N)$ . Also, if  $t_i$  is the particular parameter corresponding to  $V$ , then  $\mathcal{O}(\overline{(B_{\beta,V})_L})$  can be identified via the gluing map with  $\mathbb{F}_L[t_i^{-1}]$ . So  $\Delta^*$  is surjective, as  $\mathcal{O}(\overline{Z \cap Z'}) = \mathbb{F}_L[t_i, t_i^{-1}]$ . Thus,  $C$  is separated over  $K$  [BGR, Def. 9.6.1/1], and hence proper [BGR, Def. 9.6.2/2]. Therefore,  $C$  is an algebraic curve by the Riemann Existence Theorem.  $\square$

When a wide open  $W$  is completed to a curve  $C$  as above, the underlying affinoid  $X$  is the complement in  $C$  of a finite union of open disks. As we will now show, this results in a close connection between the reductions of  $C$  and the canonical reduction of  $X$ . Of particular interest will be the case when  $(W, X)$  is basic (defined below), in which case, provided  $K$  is stable and assuming Hypothesis T, we show that  $X$  is the minimal underlying affinoid and  $C$  has a model which reduces to  $\bar{X}^c$ .

**Lemma 2.19.** *Assume Hypothesis T. Suppose  $C$  is a smooth complete curve over  $K$ , and  $Z$  is a non-empty subset of  $C(\bar{K})$  which is Galois stable over  $K$  and open in the canonical topology (see [BGR, §7.2.1]). If  $Q$  is a point in  $C(K)$ , there exists a function  $f$  on  $C$ , defined over  $K$ , with a pole only at  $Q$  and zeroes only in  $Z$ .*

*Proof.* We can assume  $g = g(C) > 0$  and  $Q \notin Z$ . Identify  $C$  with its image in its Jacobian  $J$  by  $x \mapsto (x) - (Q)$ . Then  $U := [g]_J Z = Z[+]_J \cdots [+]_J Z$  is open in  $J(\bar{K})$ . Let  $P \neq 0 \in U$ . We claim that there is a sequence,  $m_1, m_2, \dots$ , of positive integers such that  $[m_n]_J P \rightarrow 0$ .

By Theorems 5.1, 6.6, 7.4 and 7.5 of [BL2] (see also Theorems 2.1 and 2.2 of [Ch]), there is a finite extension  $L$  of  $K$ ,<sup>3</sup> and a commutative rigid analytic group  $\hat{J}$  and formal analytic groups [Bo4, Def. 1.4] (see also the introduction to [Bo2] and [Ch, p. 397])  $J^{fm}$  and  $B$  over  $L$ , such that  $B$  is proper and we have an injective composition  $(J^{fm})^{rg} \rightarrow \hat{J} \rightarrow J_L^{rg}$  such that the image of  $(J^{fm})^{rg}$  in  $J_L^{rg}$  is a maximal connected subgroup with a formal analytic structure.<sup>4</sup> Moreover, there is a diagram with exact rows and columns

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \downarrow & & & \\
& & & \mathbb{Z}^t & & & \\
& & & \downarrow & & & \\
0 & \rightarrow & (\mathbf{G}_m^{rg})^t & \rightarrow & \hat{J} & \rightarrow & B^{rg} \rightarrow 0 \\
& & & & \downarrow & & \\
& & & & J_L^{rg} & & \\
& & & & \downarrow & & \\
& & & & 0 & & 
\end{array}$$

where  $t \in \mathbb{N}$  (the toric rank) and the image of  $\mathbb{Z}^t$  is a discrete closed subgroup. This induces an exact sequence of formal analytic groups<sup>5</sup>

$$0 \rightarrow (\mathbf{G}_m^{fm})^t \rightarrow J^{fm} \rightarrow B \rightarrow 0$$

and implies that  $\hat{J}(L)/J^{fm}(L)$  is isomorphic to  $(\mathbf{G}_m^{rg}(L)/\mathbf{G}_m^{fm}(L))^t$  and the reduction of  $J^{fm}$  over the residue field of  $L$  is semi-Abelian.<sup>6</sup>

<sup>3</sup>While the field is assumed to be algebraically closed in [BL2], it is explained on page 257 of [BL2] how to show that  $\hat{J}$  may be defined over a finite extension.

<sup>4</sup>If  $Y$  is a scheme or formal analytic space,  $Y^{rg}$  will denote the associated rigid space.

<sup>5</sup> $\mathbf{G}_m^{fm}$  denotes the formal completion of  $\mathbf{G}_m$  along its reduction.

<sup>6</sup>Formal analytic spaces have canonical reductions.

So  $J(L)/J^{f^m}(L)$  is isomorphic to  $(\mathbf{G}_m^{rg}(L)/\mathbf{G}_m^{fm}(L))^t/\Gamma$ , where  $\Gamma$  is the injective image of  $\mathbb{Z}^t \rightarrow \hat{J}(L)/J^{f^m}(L) \rightarrow (\mathbf{G}_m^{rg}(L)/\mathbf{G}_m^{fm}(L))^t$ . Assuming Hypothesis T for  $L$ ,  $\mathbf{G}_m^{rg}(L)/\mathbf{G}_m^{fm}(L) = L^*/R_L^*$  is isomorphic to a subgroup of  $\mathbb{Q}$ , and hence it follows that  $J(L)/J^{f^m}(L)$  is torsion. As all elements on a semi-Abelian variety over a finite field are torsion, some multiple  $[k]_J P$  of  $P$  lies in the image of the kernel of reduction of  $J^{f^m}$ , and then  $[p^n k]_J P \rightarrow 0$ .

Now, since  $U$  is open and  $[m_n]_J P \rightarrow 0$ , there is a positive integer  $m$  such that  $-[m-1]_J P = P[-]_J [m]_J P \in U$ . Thus  $0 \in [mg]_J Z$ . More specifically, there is a principal divisor  $D$  of the form

$$(m-1) \sum_{i=1}^g (P_i) + \sum_{i=1}^g (R_i) - mg(Q),$$

where  $P_i$  and  $R_i \in Z$ . If  $g$  is a function over  $L$  with this divisor, we can take  $f = \prod_{\sigma} g^{\sigma}$ , where  $\sigma$  ranges over embeddings of  $L/K$  into  $\mathbf{C}/K$ .  $\square$

**Lemma 2.20.** *Suppose  $C$  is a complete curve over  $K$  and  $U$  is an open disk in  $C$ . Then  $Y := C \setminus U$  is a non-empty open in the canonical topology.*

*Proof.* Let  $P$  be any point in  $Y$ , which is non-empty since  $U$  is not proper and so cannot equal  $C$ . By Riemann-Roch, we can choose a meromorphic function  $g$  on  $C$  with a pole only at  $P$ . Then because  $g|_U$  is holomorphic and finite to one,  $g(U)$  is an open disk in  $\mathbf{A}^1$ . Let  $E$  be an open disk around infinity in  $\mathbf{P}^1 \setminus g(U)$ . Then  $g^{-1}(E)$  is an open neighborhood of  $P$  in  $Y$ , in the canonical topology.  $\square$

**Proposition 2.21.** *Assume Hypothesis T. Suppose  $C$  is a smooth complete curve over  $K$ . Let  $L$  be a finite Galois extension of  $K$ , and  $T$  a finite, non-empty, Galois stable subset of  $C(L)$ . Suppose  $\mathcal{D} = \{D_t : t \in T\}$  is a Galois stable collection of disjoint open disks over  $L$  in  $C$ , such that  $D_t \cap T = \{t\}$  for all  $t \in T$ . Then if  $U = \bigcup \mathcal{D}$ ,  $X := C \setminus U$  is a one-dimensional affinoid over  $K$ , and the image of the ring of algebraic functions,  $\mathcal{O}_C(C \setminus T)$ , is dense in  $A(X)$ .*

*Proof.*  $X$  is non-empty, as  $U$  is not proper, and  $X$  is open in the canonical topology by Lemma 2.20. Therefore, Lemma 2.19 implies that for each Galois orbit  $S \subseteq T$  there exists a function  $f_S \in \mathcal{O}_C(C \setminus S)$ , defined over  $K$ , which has a pole at each  $s \in S$  and zeroes only on  $X$ . Set  $\mathcal{D}_0 = \mathcal{D}$ ,  $X_0 = X$ , and  $U_0 = U$ . Then for each  $n \geq 1$ , choose a Galois stable collection  $\mathcal{D}_n$  of  $|T|$  open disks over  $L$  in  $C$ , such that  $\mathcal{D}_{n+1} \subseteq \mathcal{D}_n$  for all  $n \geq 0$  and  $\bigcap_n \mathcal{D}_n = T$ . Set  $U_n = \bigcup \mathcal{D}_n$ , and  $X_n = C \setminus U_n$ . Let  $D_{t_n}$  be the disk in  $\mathcal{D}_n$  which contains any particular  $t \in T$ , and for any Galois orbit  $S \subseteq T$  set  $M_{S_n} = \inf\{|f_S(x)| : x \in \bigcup_{s \in S} D_{s_n}\}$ . (Note that this infimum is positive and does not belong to the set, as  $|g|_{sup}$  exists and is not equal to  $|g(x)|$  for any  $x \in D$ , when  $g$  is a rigid function on an open disk  $D$  which vanishes at only finitely many points.) Claim:

$$X_n = Z_n := \{x \in C : |f_S(x)| \leq M_{S_n}, \text{ for all Galois orbits } S \subseteq T\}.$$

It is clear that  $Z_n \subseteq X_n$ , since  $Z_n$  can not intersect  $D_{t_n}$  for any  $t \in T$ . For the other direction, note that  $f_S$  is defined over  $K$  and has poles only on  $S$ , and

so  $f_S: C \rightarrow \mathbf{P}^1$  has degree  $|S|d_S$  where  $d_S := -\text{ord}_s f_S$  for any  $s \in S$ . Moreover, as  $f_S$  has no zeroes on  $\mathcal{D}$ ,  $f_S|_{D_{s_n}}$  is a  $d_S$  to 1 map onto the disk,  $\mathbf{P}^1 \setminus B_K[M_{S_n}]$ . It follows that  $M_{S_n} \in \mathcal{R}$  and  $\|f_S\|_{X_n} = M_{S_n}$ . Thus,  $X_n \subseteq Z_n$ . So  $X_n = Z_n$ , and in particular  $X_n$  is an affinoid.

Now, for each  $n$  and  $S$ , we may choose  $e_{S_n} \in \mathbb{N}$  and  $a_{S_n} \in K^*$  such that  $|a_{S_n}| = M_{S_n}^{e_{S_n}}$ . Then, using the notation of [BGR, §7.2.3], we have

$$Z_n = Z_{n+1}(f_S^{e_{S_n}}/a_{S_n} : S \text{ is a Galois orbit in } T).$$

It follows from Proposition 7.2.3/1 of [BGR] that the image of  $A(X_{n+1})$  is dense in  $A(X_n)$ . Suppose  $g \in A(X)$  and  $\epsilon > 0$ . Then there exist functions  $h_n \in A(X_n)$  such that  $\|h_1 - g\|_X < \epsilon$  and  $\|h_{n+1} - h_n\|_{X_n} < \epsilon/n$  for  $n \geq 1$ . It follows that the sequence  $h_n$  converges to an element  $h_\epsilon \in A(C \setminus T)$  such that  $\|h_\epsilon - g\|_X < \epsilon$ . The proposition follows from the fact that  $\mathcal{O}_C(C \setminus T)$  is dense in  $A(C \setminus T)$ .  $\square$

**Corollary 2.22.** *Assume Hypothesis T. Let  $W$  be a wide open over  $K$ . Then the image of  $A(W)$  is dense in  $A(X)$  for each underlying affinoid  $X$ .*

*Proof.* Glue open disks  $B_V$  to  $W$  to make a complete curve  $C$  as in the proof of Theorem 2.18. For each  $V \in \mathcal{S}$ , choose a point  $t_V \in B_V \setminus W$  which is defined over  $K$ . Then let  $T = \{t_V : V \in \mathcal{S}\}$  and follow the above procedure, noting that the map from  $\mathcal{O}_C(C \setminus T)$  to  $A(X)$  factors through  $A(W)$ .  $\square$

**Corollary 2.23.** *With the same hypotheses and notation as Proposition 2.21, set  $A_0 = \{f \in \mathcal{O}_C(C \setminus T) : \|f\|_X \leq 1\}$ . Suppose  $A_0 \otimes \mathbb{F}_K$  is reduced. Then  $\text{Spec } A_0 \otimes \mathbb{F}_K \cong \overline{X}$ .*

A **basic wide open pair** over  $K$  is a pair  $(W, X)$  where  $W$  is a connected wide open over  $K$  and  $X$  is an underlying affinoid. In addition, we require that  $\|A(X)\|_X = |K|$ , that  $X$  has irreducible reduction with at worst ordinary double points as singularities, and that the components of  $W \setminus X$  are isomorphic to annuli of the form  $A_K(1, s)$ . If  $(W, X)$  is a basic wide open pair for some  $X$ , we say that  $W$  is a **basic wide open**. By Proposition 2.21 and Corollary 2.23, basic wide open pairs can be constructed by taking  $(W, X) = (C \setminus \bigcup_{i=1}^n D_i, C \setminus \bigcup_{i=1}^n U_i)$ . Here  $C$  is a connected smooth complete curve over  $K$ , which has a model  $\mathcal{C}$  over  $R_K$  whose reduction is irreducible and has at worst ordinary double points as singularities,  $\{U_1, \dots, U_n\}$  is a finite collection of distinct residue classes of smooth points, and each  $D_i$  is an affinoid disk in  $U_i$ . The converse, that all basic wide open pairs can be constructed in this manner, follows, when  $K$  is stable and assuming Hypothesis T, from Theorem 2.27 (and thus the two notions are equivalent in this case).

**Lemma 2.24.** *Assume Hypothesis T. Suppose  $f : X \rightarrow Y$  is a map between smooth one dimensional affinoids over  $K$ , and  $\overline{X}$  is irreducible.*

- (i) *If  $\bar{f} : \overline{X} \rightarrow \overline{Y}$  is a surjection, then  $f$  is a surjection.*
- (ii) *If  $\bar{f}(\overline{X}) \subseteq \overline{Y}$  is an open affine and  $X(\mathbf{C}) \rightarrow Y(\mathbf{C})$  is an injection, then  $f$  is an injection.*

*Proof.* For both parts, we may extend scalars to  $\mathbf{C}$ . To prove (i), suppose  $\bar{f}$  is a surjection but that there exists a  $y \in Y$  which is not in the image of  $f$ . Let  $\lambda \in A^o(Y)$  be a function which vanishes only at  $y$  and such that  $\|\lambda\|_Y = 1$ .<sup>7</sup> On the one hand, if we let  $L = f^*(\lambda)$ , the fact that  $\bar{f}$  is a surjection implies that  $\|L\|_X = 1$ . On the other hand, as  $L$  does not vanish on  $X$ ,  $L^{-1}$  exists and we may choose  $c \in \mathbf{C}$  such that  $|c| = \|L^{-1}\|_X$ . Now the fact that  $\bar{f}$  is a surjection implies that  $|c| > 1$ . Thus, if we let  $M = c^{-1}L^{-1}$ , we have  $\bar{L}, \bar{M} \neq 0$  but  $\bar{L}\bar{M} = 0$ . So  $\bar{X}$  must be reducible.

For (ii), let  $Y' \subseteq Y$  be the Zariski subaffinoid for which  $\bar{f}(\bar{X}) = \bar{Y}'$ . Suppose there are distinct points  $x_1, x_2 \in \bar{X}$  for which  $\bar{f}(x_1) = \bar{f}(x_2)$ . Let  $X' = X \setminus R(x_2)$ . Then  $f$  restricts to a map  $f' : X' \rightarrow Y'$  which reduces to a surjection. Thus, by (i),  $f'$  is a surjection. But this is a contradiction since  $f(R(x_2)) \subseteq Y'$  and  $f$  is an injection. Therefore,  $\bar{f}$  must also be an injection.  $\square$

**Lemma 2.25.** *Suppose  $h: B \rightarrow Y$  is an analytic map from an open annulus or open disk into a reduced affinoid. Then the image of  $B$  is contained in a residue class of  $Y$ .*

*Proof.* This is clear when  $Y$  is an affinoid disk. The general case follows.  $\square$

**Remark 2.26.** *The same statement is true with  $B$  a connected wide open in place of an open annulus.*

**Theorem 2.27.** *Suppose  $K$  is stable and satisfies Hypothesis T. Let  $(W, X)$  be a basic wide open pair over  $K$ . Attach disks  $B_V$  to  $W$  to obtain a complete curve  $C$ , as in the proof of Theorem 2.18. Then  $C$  has a model over  $R_K$  whose reduction is  $\bar{X}^c$ . Moreover, if  $x$  is a point at  $\infty$  in  $\bar{X}^c(\mathbb{F})$ , then  $x \in \bar{X}^c(\mathbb{F}_K)$  and  $\{P \in C(\mathbf{C}) : \bar{P} = x\}$  is equal to  $B_V(\mathbf{C})$  for some  $V \in \mathcal{S} = CC(W \setminus X)$ .*

*Proof.* Choose a finite, Galois stable set of points  $Y \subset X(L)$ , for some finite extension  $L$  of  $K$ , which reduce to distinct smooth points in  $\bar{X}_L(\mathbb{F}_L)$ . The set of residue classes of  $X_L$ ,  $\{R(\bar{y}) : y \in Y\}$ , is a Galois stable set of open disks in  $C$  over  $L$ . Therefore, by Proposition 2.21,  $Z := C \setminus \bigcup_{y \in Y} R(\bar{y})$  is an affinoid over  $K$ . Moreover,  $X_1 := X \cap Z$  is a formal subdomain of  $X$  (see [BL1, pg. 351]), whose reduction is  $\bar{X} \setminus \{\bar{y} : y \in Y\}$ . We will show that  $X_1$  is also a formal subdomain of  $Z$ , and hence  $\mathcal{C} := \{X, Z\}$  is a formal covering of  $C$ .

To do this, let  $Z_{\mathcal{T}} := Z \setminus \bigcup_{V \in \mathcal{T}} B_V$  for any  $\mathcal{T} \subseteq \mathcal{S}$ . This is an affinoid over  $K$  by Proposition 2.21. We claim that  $Z_{\mathcal{T}}$  has irreducible reduction, and that  $B_V$  is a residue class of  $Z_{\mathcal{T}}$  for each  $V \in \mathcal{S} \setminus \mathcal{T}$ . This is clearly true for  $\mathcal{T} = \mathcal{S}$ , since  $Z_{\mathcal{S}} = X \setminus \bigcup_{y \in Y} R(\bar{y})$  is a Zariski subaffinoid of  $X$  and  $\mathcal{S} \setminus \mathcal{T}$  is empty. Suppose it holds for some  $\mathcal{T}$  and that  $V \in \mathcal{T}$ . Let  $\mathcal{T}' = \mathcal{T} \setminus \{V\}$ , so that  $Z_{\mathcal{T}'} = Z_{\mathcal{T}} \coprod B_V$ . By Lemma 2.25, applied to the inclusion of  $B_V$  into  $Z_{\mathcal{T}'}$ ,  $B_V$  is contained in the residue class  $R(\bar{t}_V)$ . If  $B_V \neq R(\bar{t}_V)$ , the map  $\bar{Z}_{\mathcal{T}} \rightarrow \bar{Z}_{\mathcal{T}'}$  is a surjection. But this is impossible by Lemma 2.24 since  $Z_{\mathcal{T}}$  has irreducible reduction and  $Z_{\mathcal{T}} \neq Z_{\mathcal{T}'}$ . Therefore,  $B_V$  is a residue class of  $Z_{\mathcal{T}'}$ ,  $Z_{\mathcal{T}}$  is a Zariski subaffinoid of

<sup>7</sup>This can be done by first embedding  $Y$  in a smooth, complete curve (see [vP1, Theorem 1.1]) and then applying Lemma 2.19.

$Z_{\mathcal{T}'}$ , and in particular  $Z_{\mathcal{T}'}$  has irreducible reduction. The claim now follows for all  $\mathcal{T}$  by induction. Taking  $\mathcal{T} = \emptyset$ , we see that  $Z$  has irreducible reduction, and that each disk,  $B_V$ , is a residue class of  $Z$ . Thus,  $X_1$  is a formal subdomain of  $Z$ , and  $\mathcal{C}$  is a formal covering of  $C$ . Moreover, by Proposition 2.8, the reduction of  $Z$  is the disjoint union of  $\overline{X_1}$  and finitely many smooth points. Thus  $C$  has semi-stable reduction with respect to  $\mathcal{C}$  (see [BL1, Def. 1.5]). So using the argument of [BL1, pg. 377], it follows that  $C$  has a model with reduction  $\overline{X^c}$ . Moreover, the residue classes of the “points at infinity” on  $\overline{X^c}$  are precisely the disks  $B_V$  over  $K$ .  $\square$

It may be proven that over  $\mathbf{C}$ , all wide opens which are not disks or annuli have minimal underlying affinoids. In fact, one can show that if  $W$  is a wide open over  $K$  which is not a disk or an annulus,  $W$  has an affinoid subdomain which becomes the minimal underlying affinoid of  $W_L$  where  $L$  is a finite extension of  $K$  (see Remark 2.41). However, this fact is not used in this paper.

**Lemma 2.28.** *Suppose  $K$  is stable and assume Hypothesis T. If  $(W, Z)$  is a basic wide open pair over  $K$ , and  $W$  is not a disk or annulus, then  $Z$  is a minimal underlying affinoid of  $W$ .*

*Proof.* Suppose there are  $e$  ends. Glue in disks, as above, to get a smooth complete curve  $C$ , so that  $C \setminus Z$  is the union of  $e$  open disks,  $U_1 \dots U_e$ . Then by Theorem 2.27,  $C$  will have a model  $\mathcal{C}$  with reduction isomorphic to the completion of  $Z$ .

We can and will extend scalars to  $\mathbf{C}$ . Suppose  $V$  is any underlying affinoid of  $W$  and  $A$  is a component of  $W \setminus V$ . Then  $A \cap U_i \neq \emptyset$  for some  $i$ . Set  $U = U_i$ . We claim that  $A$  is contained in  $U$ .

Identify  $A$  with  $A_{\mathbf{C}}(r, s)$  so that  $A_{\mathbf{C}}(t, s)$  is connected to  $V$  for any  $t \in \mathcal{R}(r, s)$ . It follows from Proposition 5.4(c) of [BL1] that every circle in  $A$  which intersects a residue class of  $C$  is contained in that class. Hence  $A \cap U$  contains  $C_{\mathbf{C}}[t]$  for any  $t \in \mathcal{R}(r, s)$  with  $C_{\mathbf{C}}[t] \cap U \neq \emptyset$ . In fact,  $A \cap U \supset A_{\mathbf{C}}(r, t)$  for any such  $t$ . Let  $q = LUB\{t \in \mathcal{R}(r, s) : A_{\mathbf{C}}(r, t) \subseteq U\}$ . Suppose that  $q < s$ , and let

$$\begin{aligned} v &= LUB\{t \in \mathcal{R}[q, s] : C_{\mathbf{C}}[t] \cap Z \neq \emptyset\} \\ &= GLB\{t \in \mathcal{R}[q, s] : C_{\mathbf{C}}[t] \cap Z = \emptyset\}. \end{aligned}$$

The number  $v$  exists since  $U$  is disconnected from  $U_j$ ,  $j \neq i$ . For  $u \in \mathcal{R}[q, v)$ ,  $C_{\mathbf{C}}[u] \subseteq Z$  (again, by [BL1, Prop. 5.4(c)]). Let  $w = q$  if  $q \in \mathcal{R}$ , and  $w \in \mathcal{R}[q, u)$  otherwise. Then set  $Y = A_{\mathbf{C}}[w, u]$ . We have a rigid morphism  $Y \rightarrow Z$ . Since  $\overline{Y}$  is either a line or two lines crossing at a point, and  $\overline{Z}$  is irreducible, not isomorphic to  $\mathbf{A}^1$  or  $\mathbf{G}_m$ , with only ordinary double points as singularities, it follows that the map  $\overline{Y} \rightarrow \overline{Z}$  is constant. This means  $A \setminus U$  is contained in a residue class  $R$  of  $Z$ . Thus  $\{U, R\}$  is a disjoint admissible cover of  $A$ . This is impossible as  $A$  is connected.

From the contradiction, we know that  $q = s$ , and thus  $A \subseteq U$ . Now, since each component of  $W \setminus V$  is contained in  $W \setminus Z$ , we have shown that  $Z \subseteq V$ .  $\square$

The final two results of this section provide useful criteria for determining when a rigid space is a wide open.

**Theorem 2.29.** *Suppose  $X$  is a smooth one dimensional affinoid over a stable field  $K$  satisfying Hypothesis B, and  $x$  is a point of degree one on  $\overline{X}$ . Then, if  $U = R_X(x)$ , there is a finite extension  $L$  of  $K$  such that  $U_L$  is a connected wide open over  $L$ . Moreover, the number of ends of  $U_L$  equals the number of branches of  $\overline{X}_L$  through  $x$ .*

This is a consequence of the following lemma. Recall that  $\mathbb{F}_K \subseteq \overline{\mathbb{F}}$ .

**Lemma 2.30.** *Suppose  $X$  is a pure one dimensional reduced affinoid over a stable field  $K$  satisfying Hypothesis B, with reduced reduction, and  $x \in \overline{X}(\mathbb{F}_K)$  is a degree one point. Choose any  $f \in A^\circ(X)$  such that  $\bar{f}$  has an isolated zero at  $x$  (i.e.,  $x$  is the only zero in a Zariski neighborhood). For  $r \in \mathcal{R}(0, 1)$  let  $V(r)$  be the subspace of  $X$  such that*

$$V(r)(\mathbf{C}) = \{y \in R(x)(\mathbf{C}) : r < |f(y)| < 1\}.$$

*Then for  $r$  sufficiently close to 1 there is a finite extension  $L$  of  $K$  such that  $V_L(r)$  is a disjoint union of  $m := |(n^{-1}x)(\overline{\mathbb{F}})|$  open annuli, where  $n: Y \rightarrow X_{\overline{\mathbb{F}}}$  is the normalization of  $X_{\overline{\mathbb{F}}} := \overline{X} \otimes_{\mathbb{F}_K} \overline{\mathbb{F}}$ .*

*Proof.* Without loss of generality, we may assume that  $x$  is the only zero of  $\bar{f}$  (otherwise replace  $X$  with a suitable Zariski subaffinoid). Let  $Z := Z_r$  be the subaffinoid of  $X$  whose  $\mathbf{C}$  valued points are  $\{y \in X(\mathbf{C}) : |f(y)| \geq r\}$ . Let  $X_x$  be the curve which is obtained from  $Y$  by identifying  $n^{-1}(x')$  to a point for each  $x' \in X_{\overline{\mathbb{F}}}(\overline{\mathbb{F}}) \setminus \{x\}$  (thus,  $X_x$  is the minimal finite surjective cover of  $X_{\overline{\mathbb{F}}}$  which is smooth at all points above  $x$ ). It is proven in [BL1] (see remark after Lemma 2.4) that for  $r \in \mathcal{R}(0, 1)$  sufficiently close to 1, the reduction of  $Z_{\mathbf{C}}$  is isomorphic to the union of  $X_x$  and  $m$  lines, each crossing a single point above  $x$  normally.<sup>8</sup>

Now, there is a finite extension  $M$  of  $K$  such that  $\overline{Z}_M$  is reduced, and then  $\overline{Z}_{\mathbf{C}} \cong (\overline{Z}_M)_{\overline{\mathbb{F}}}$ . Thus, there is a finite extension  $L$  of  $K$  such that  $\overline{Z}_L$  is isomorphic to the union of a finite surjective cover of  $\overline{X}_{\overline{\mathbb{F}}_L}$  which is smooth at all points above  $x$ , and  $m$  lines each crossing a single point above  $x$  normally. Now apply Proposition 2.10.  $\square$

**Proposition 2.31.** *Assume Hypothesis T. Suppose  $f: U \rightarrow W$  is a finite surjective morphism over  $K$  of a smooth one dimensional rigid space onto a wide open, with finitely many branch points all defined over  $K$ . If  $f$  has degree strictly less than  $p$ , then  $U$  is a wide open over  $K$ .*

*Proof.* First we claim that an underlying affinoid  $X \subseteq W$  can be chosen so that it contains the set  $\mathcal{B}$  of branch points of  $f$ . Indeed, let  $X_1$  be any underlying affinoid of  $W$ . Glue disks  $B_V$  onto  $W$  for each annulus,  $V \in W \setminus X_1$ , as in the proof of Theorem 2.18, to obtain a complete curve  $C$ . Then for each  $V$ , choose an open disk  $D_V$  over  $K$  such that  $B_V \setminus \alpha_V(V) \subset D_V \subseteq B_V$  and  $D_V \cap \mathcal{B} = \emptyset$ .

<sup>8</sup>This is a minor correction of the statement in [BL1].

The rigid subspace  $X := C \setminus \bigcup D_V$  is an affinoid by Proposition 2.21 which is disjoint from  $\mathcal{B}$  and easily shown to be underlying in  $W$ .

Now suppose  $X$  is relatively compact in some affinoid  $Y \subseteq W$ . Then  $f^{-1}(X)$  and  $f^{-1}(Y)$  are affinoids in  $U$ . Moreover, as  $f$  is finite, and the image of  $\overline{X}$  in  $\overline{Y}$  is finite, it follows that the image of  $\overline{f^{-1}(X)}$  in  $\overline{f^{-1}(Y)}$  is finite. So  $f^{-1}(X)$  is relatively compact in  $f^{-1}(Y)$ . All that remains to check is that  $U \setminus f^{-1}(X)$  is the disjoint union of open annuli, and for this Theorem 2.6 suffices.  $\square$

### 2.3 Semi-stable Coverings

For a wide open  $W$  over  $K$ , let  $H_{DR}^i(W/K) = D^i(W/K)$ . Using Lemma 2.13, the arguments in the proof of [C1, Thm. 4.2] generalize and allow us to conclude that  $H_{DR}^i(W/K)$  is finite dimensional over  $K$ . We define the genus of  $W$ , which we denote by  $g(W)$ , to be

$$\frac{1}{2}(\dim_K H_{DR}^1(W/K) - e(W) + 1).$$

Then  $2g(W)$  can be interpreted as the dimension of the “first compactly supported de Rham cohomology group” of  $W$ . For example, in Corollary 2.33, we show that

$$2g(W) = \dim_K(\ker(H_{DR}^1(W/K) \rightarrow H_{DR}^1((W \setminus X)/K))),$$

where  $X$  is any underlying affinoid of  $W$ . We also show in Proposition 2.32 that if a wide open  $W$  is completed to a projective curve  $C$  by attaching disks at the ends, as in Theorem 2.18, then  $g(W) = g(C)$ . As an immediate corollary of this and Theorem 2.27, when  $(W, X)$  is a basic wide open pair over a complete, stable field  $K$  satisfying Hypothesis T, and  $X$  has good reduction  $\overline{X}$ ,  $(\overline{X})^c$  will also have genus  $g(W)$ .

**Proposition 2.32.** *Let  $W$  be a connected wide open over  $K$ . Suppose  $C$  is a smooth, complete curve (over  $K$ ) obtained by attaching disks at the ends of  $W$ , as in Theorem 2.18. Then  $g(W) = g(C)$ .*

*Proof.* The main idea is to view  $W$  and the attached disks as an admissible covering of  $C$ , and then to apply the (generalized) Mayer-Vietoris sequence of de Rham cohomology (over  $K$ ). So first suppose  $D_1, \dots, D_n$  are the disks, and set  $D_0 = W$ . Then Mayer-Vietoris gives us the following exact sequence.

$$\begin{aligned} 0 \rightarrow H_{DR}^0(C) \rightarrow \bigoplus_i H_{DR}^0(D_i) \rightarrow \bigoplus_{i \neq j} H_{DR}^0(D_i \cap D_j) \rightarrow \\ H_{DR}^1(C) \rightarrow \bigoplus_i H_{DR}^1(D_i) \rightarrow \bigoplus_{i \neq j} H_{DR}^1(D_i \cap D_j) \rightarrow H_{DR}^2(C) \rightarrow 0 \end{aligned}$$

Now using Lemma 2.13, the above definition of  $g(W)$ , and the fact that  $H_{DR}^1(D_i) = 0$  for  $i > 0$ , we count dimensions to obtain:

$$1 - (e(W) + 1) + e(W) - 2g(C) + (2g(W) + e(W) - 1) - e(W) + 1 = 0.$$

From this we conclude that  $g(C) = g(W)$ .  $\square$

**Corollary 2.33.** *Suppose  $W$  is a wide open over  $K$  and  $X$  is an underlying affinoid of  $W$ . Then*

$$2g(W) = \dim_K(\ker(H_{DR}^1(W/K) \rightarrow H_{DR}^1((W \setminus X)/K))).$$

*Proof.* Suppose  $C$  is a smooth complete curve obtained by gluing disks to the ends of  $W$ . Then arguing from Mayer-Vietoris exactly as in the above proof, we have an exact sequence:

$$0 \rightarrow H_{DR}^1(C) \rightarrow H_{DR}^1(W) \rightarrow H_{DR}^1(W \setminus X) \rightarrow K \rightarrow 0.$$

Now apply the previous proposition. □

Let  $C$  be a wide open or a smooth proper curve over  $K$ . Let  $\mathcal{C}$  be a finite collection of basic wide open pairs  $(U, U^u)$  over  $K$  such that  $\mathcal{C}^w := \{U, (U, U^u) \in \mathcal{C}\}$  is an admissible covering of  $C$ . Then we call  $\mathcal{C}$  a **semi-stable covering** over  $K$  if the following conditions hold:

- (i) if  $U, V \in \mathcal{C}^w$  and  $U \neq V$ ,  $U \cap V$  is a disjoint union of connected components of  $U \setminus U^u$  (by definition, annuli of the form  $A_K(1, s)$ ),
- (ii) if  $U, V$  and  $W$  are three distinct elements of  $\mathcal{C}^w$ ,  $U \cap V \cap W = \emptyset$ .

We say that a semi-stable covering  $\mathcal{C}$  is **stable** if none of the elements of  $\mathcal{C}^w$  are disks or annuli. Having a semi-stable covering is not immediately equivalent to having a “semi-stable reduction” in the sense of [BL1, Definition 1.5]. When the context is clear, we will abuse notation by dropping the superscript  $w$  and writing  $U \in \mathcal{C}$  to mean  $U \in \mathcal{C}^w$ .

**Proposition 2.34.** *Suppose  $\mathcal{C}$  is a semi-stable covering of a smooth proper curve  $C$  over  $K$ . Let  $\Gamma_{\mathcal{C}}$  be the unoriented graph without loops, whose vertices correspond to the elements of  $\mathcal{C}$ , and whose edges with endpoints corresponding to distinct  $U, V \in \mathcal{C}$  correspond to the connected components of  $U \cap V$ . Then*

$$g(C) = \sum_{U \in \mathcal{C}} g(U) + \text{Betti}(\Gamma_{\mathcal{C}}).$$

*Proof.* Again, we begin with the Mayer-Vietoris sequence (of de Rham cohomology over  $K$ ) associated to this covering.

$$\cdots \rightarrow \bigoplus_{U, V \in \mathcal{C}} H_{DR}^{i-1}(U \cap V) \rightarrow H_{DR}^i(C) \rightarrow \bigoplus_{U \in \mathcal{C}} H_{DR}^i(U) \rightarrow \cdots$$

It is immediate that  $H_{DR}^0(C) \cong H_{DR}^2(C) \cong K$ ,  $\bigoplus_{U \in \mathcal{C}} H_{DR}^2(U) = 0$ , and  $\bigoplus_{U \in \mathcal{C}} H_{DR}^0(U) \cong K^{\#\mathcal{C}}$ . Also, by applying Lemma 2.13 and condition (i) from above, we see that

$$\bigoplus_{U, V \in \mathcal{C}} H_{DR}^0(U \cap V) \cong \bigoplus_{U, V \in \mathcal{C}} H_{DR}^1(U \cap V) \cong K^{\#\mathcal{E}},$$

where  $\mathcal{E}$  is the edge set of  $\Gamma_C$ . Now to prove the proposition, we simply count dimensions over  $K$  and compute the dimension of  $H_{DR}^1(C)$  using the exact sequence. We have,

$$\begin{aligned} 2g(C) &= \sum_{U \in \mathcal{C}} (2g(U) + e(U) - 1) - \#\mathcal{C} + 2 \\ &= 2\left(\sum_{U \in \mathcal{C}} g(U) + \#\mathcal{E} - \#\mathcal{C} + 1\right) \\ &= 2\left(\sum_{U \in \mathcal{C}} g(U) + \text{Betti}(\Gamma_C)\right). \end{aligned}$$

□

**Definition 2.35.** A semi-stable model  $\mathcal{B}$  of a curve  $C$  over  $K$  is a flat, proper scheme over  $R_K$  whose generic fiber is  $C$ , such that all of the singular points of the special fiber of  $\mathcal{B}$  have degree 1 and are ordinary double points. We say that  $\mathcal{B}$  is **stable** if it is the final object in the category of semi-stable models over  $K$ .<sup>9</sup>

We refer the reader to [BL1] and [vP2] for a rigid analytic treatment of the theory of stable models of curves over complete non-archimedean fields, and in particular, for a rigid-analytic proof of generalization to arbitrary complete non-archimedean fields [vP2, Corollary 3.3] (see also [BL1]<sup>10</sup>) of the existence theorem of Deligne and Mumford. Moreover, we will use the results of [BL1] and [BL2] to prove the following theorem which relates stable coverings to stable models. This result generalizes Proposition 2.1 of [C2], and the proof we will give is more complete than the one given there.

**Theorem 2.36.** Let  $C$  be a smooth complete curve over a stable field  $K$  satisfying Hypothesis B.

- (i) If  $C$  has a semi-stable model over  $R_K$  whose reduction has at least two components, then  $C$  has an associated semi-stable covering over  $K$ .
- (ii) If  $K$  satisfies Hypothesis T, and  $C$  has a semi-stable covering over  $K$ , then  $C$  has an associated semi-stable model over  $R_K$  whose reduction has at least two components.<sup>11</sup>

Stable coverings are precisely those which correspond to stable models whose reductions have at least two components.

<sup>9</sup>This weakens the definition of semi-stable model in [Mu] since it allows smooth rational components which meet the other components in only one point. Requiring the singular points to have degree 1 means that  $X_0(p)$  usually does not have a stable model over  $\mathbb{Q}_p$ , but does over  $W(\mathbb{F}_{p^2}) \otimes \mathbb{Q}_p$ .

<sup>10</sup>While proving the theorem of Deligne and Mumford on page 377, the authors of [BL1] remark that their argument does not require the field to be discretely valued.

<sup>11</sup>In fact, we have a natural one-to-one correspondence between semi-stable coverings and semi-stable models whose reductions have at least two components.

*Proof.* Suppose  $\mathcal{C}$  is a semi-stable model for  $C$  over  $R_K$ , and let  $\mathcal{I}_{\mathcal{C}}$  be the set of irreducible components in the reduction of  $\mathcal{C}$ . For each  $\Gamma \in \mathcal{I}_{\mathcal{C}}$ , let

$$\Gamma^\circ = \Gamma \setminus \bigcup_{\substack{\Gamma' \in \mathcal{I}_{\mathcal{C}} \\ \Gamma' \neq \Gamma}} \Gamma'.$$

Assume, without loss of generality, that  $\bar{\mathcal{C}}$  is connected.

For each affine open  $U \subseteq \bar{\mathcal{C}}$ , there is a natural affinoid subdomain of  $C^{rg}$ , which we denote  $\text{Red}^{-1}U$ , whose points are all the points of  $C^{rg}$  which reduce to points of  $U$ . To see this, let  $\text{Spec } S$  be any affine open subscheme of  $\mathcal{C}$  which reduces to  $U$  and  $\hat{S} = \varprojlim_n S/\pi^n S$  for some  $\pi \in R_K$ ,  $0 < |\pi| < 1$ . Then  $\hat{S}$  is an admissible  $R_K$ -algebra in the sense of [BL3, p. 293], as can be seen using [BL3, Lemma 1.2]. Then  $\hat{S} \otimes_{R_K} K$  is an affinoid algebra over  $K$  (see [BL3, §4]) which up to canonical isomorphism does not depend on the choices. We call the affinoid,  $\text{Sp}(\hat{S} \otimes_{R_K} K)$ ,  $\text{Red}^{-1}U$ . Because  $U$  is reduced,  $\overline{\text{Red}^{-1}U} \cong U$ . More generally, suppose  $V$  is the union of finitely many subschemes  $W$  of  $\bar{\mathcal{C}}$ , with each contained in some affine open  $U_W$ . Then we let  $\text{Red}^{-1}V$  be the open rigid subspace which is the union in  $C^{rg}$  of the subspaces,  $\text{Red}^{-1}W \subseteq \text{Red}^{-1}U_W$ , as was defined in the beginning of §2. This subspace is independent of the choices of  $U_W$ .

If  $\Gamma \in \mathcal{I}_{\mathcal{C}}$ , let  $W_\Gamma = \text{Red}^{-1}\Gamma$  and  $X_\Gamma = \text{Red}^{-1}\Gamma^\circ$ . Claim:  $\{(W_\Gamma, X_\Gamma) : \Gamma \in \mathcal{I}_{\mathcal{C}}\}$  is semi-stable covering. First,  $W_\Gamma$  is a smooth one dimensional rigid space over  $K$ , and  $X_\Gamma$  is an affinoid subdomain, such that  $W_\Gamma \setminus X_\Gamma$  is a disjoint union of a finite number of annuli of the form  $A_K(1, s)$  by Proposition 2.10. Also,  $X_\Gamma$  has absolutely irreducible reduction with at worst ordinary double points as singularities. Moreover, if  $\Gamma, \Gamma', \Gamma'' \in \mathcal{I}_{\mathcal{C}}$ ,  $W_\Gamma \cap W_{\Gamma'}$  is a union of connected components of  $W_\Gamma \setminus X_\Gamma$  if  $\Gamma \neq \Gamma'$ , and  $W_\Gamma \cap W_{\Gamma'} \cap W_{\Gamma''} = \emptyset$  if  $\Gamma, \Gamma'$  and  $\Gamma''$  are all distinct.

What remains to be shown for (i) is that  $(W_\Gamma, X_\Gamma)$  is a basic wide open pair for each  $\Gamma \in \mathcal{I}_{\mathcal{C}}$ , and for that, all we have to show is that there exists an affinoid subdomain  $Y$  of  $W_\Gamma$  such that  $X_\Gamma$  is relatively compact in  $Y$  and  $Y \cap V$  is a semi-open annulus for each connected component of  $W_\Gamma \setminus X_\Gamma$ . (That  $W_\Gamma$  is connected will follow from the absolute irreducibility of  $\bar{X}_\Gamma$ .) For this, first, let  $\mathcal{S}_\Gamma$  be the set of singular points in  $\bar{\mathcal{C}}$  where  $\Gamma$  intersects some other component. Blow up  $\mathcal{C}$  at every point in  $\mathcal{S}_\Gamma$  to obtain a new model,  $\mathcal{C}_\Gamma$ , which is defined over  $K$  and becomes semi-stable over an, at worst, quadratic extension  $L$ . Let  $\hat{\Gamma}$  be the proper transform of  $\Gamma$  in  $\mathcal{C}_\Gamma$ , and let  $\mathcal{I}_{\mathcal{C}_\Gamma}$  be the set of irreducible components in the reduction of  $\mathcal{C}_\Gamma$ . Set

$$\tilde{Y}_\Gamma = \bar{\mathcal{C}}_\Gamma \setminus \bigcup_{\substack{\Gamma' \in \mathcal{I}_{\mathcal{C}_\Gamma} \\ \Gamma' \cap \hat{\Gamma} = \emptyset}} \Gamma',$$

and let  $Y_\Gamma = \text{Red}^{-1}(\tilde{Y}_\Gamma)$ . It is clear that  $(X_\Gamma)_L \subseteq Y_\Gamma \subseteq (W_\Gamma)_L$  and  $Y_\Gamma$  is naturally defined over  $K$ . Although  $\tilde{Y}_\Gamma$  is not an affine open in  $\bar{\mathcal{C}}_\Gamma$ ,  $Y_\Gamma$  is the reduction inverse of an affine open in the model obtained from  $\mathcal{C}_\Gamma$  by blowing

down  $\hat{\Gamma}$ . This affine open will consist of  $|\mathcal{S}_\Gamma|$  lines intersecting in a single singular point which contains the reduction of  $X_\Gamma$ . Thus, not only is  $Y_\Gamma$  also an affinoid subdomain of  $W_\Gamma$ , but  $X_\Gamma$  is relatively compact in  $Y_\Gamma$ . Finally, by applying Proposition 2.10 again, we see that the intersection of  $Y_\Gamma$  with each component of  $W_\Gamma \setminus X_\Gamma$  is a semi-open annulus. Therefore, we are done with (i).

To prove (ii), suppose  $\mathcal{C}$  is a semi-stable covering of  $C$ . Then by Theorem 2.27, there is a natural one-to-one correspondence between  $(\overline{U^u})^c \setminus \overline{U^u}$ ,  $CC(U \setminus U^u)$ , and  $\mathcal{E}(U)$ , for each  $U \in \mathcal{C}$ . If  $e \in \mathcal{E}(U)$ , let  $x(e)$  denote the corresponding point on  $(\overline{U^u})^c$  and let  $A(e)$  denote the corresponding connected component of  $U \setminus U^u$  (an annulus). If  $e \in \mathcal{E}(U)$  and  $f \in \mathcal{E}(V)$ , for  $U, V \in \mathcal{C}$ , we say that  $e \sim f$  whenever  $A(e) = A(f)$  (equivalently,  $A(e) \cap A(f) \neq \emptyset$ ). Let  $\mathcal{E}$  denote the quotient of  $\coprod_{U \in \mathcal{C}} \mathcal{E}(U)$  by this equivalence relation. We define  $\bar{\mathcal{C}}$  to be the curve over  $\mathbb{F}_K$  obtained from  $\coprod_{U \in \mathcal{C}} (\overline{U^u})^c$  by identifying the points  $x(e)$  and  $x(f)$  whenever  $e \sim f$ . The reduction maps from  $U(\mathbf{C}) \rightarrow (\overline{U})^c(\mathbb{F})$  for each  $U \in \mathcal{C}$ , which are guaranteed by Theorem 2.27, patch together to form a natural Galois equivariant reduction map from  $C(\mathbf{C}) \rightarrow \bar{\mathcal{C}}(\mathbb{F})$ . We will show that in fact there is a model over  $R_K$  whose reduction is  $\bar{\mathcal{C}}$ .

Let  $T$  be a finite Galois stable set of points of  $C(\bar{K})$  which, by the above reduction map, injects into the smooth locus of  $\bar{\mathcal{C}}$ , and such that  $T \cap U \neq \emptyset$  for each  $U \in \mathcal{C}$ . Since each  $t \in T$  lies on a unique  $U^u$ , the residue class,  $R(\bar{t}) := R_{U^u}(\bar{t})$ , is well-defined and can be viewed as an open disk in  $C$  over  $\bar{K}$ . Moreover, as  $\mathcal{C}$  is defined over  $K$ ,  $R(T) := \bigcup_{t \in T} R(\bar{t})$  is Galois stable over  $K$ . So by Proposition 2.21,  $Z_T := C \setminus R(T)$  is an affinoid over  $K$ . We want to show that  $\bar{Z}_T = \bar{\mathcal{C}} \setminus \bar{T}$ , where  $\bar{T} = \{\bar{t} : t \in T\}$ .

For each  $U \in \mathcal{C}$ , we let  $U_T = U^u \setminus R(T)$ , a Zariski subaffinoid of  $U^u$ . Then the affinoid  $Z_T$  is the disjoint union of  $\coprod_{U \in \mathcal{C}} U_T$  and  $\coprod_{e \in \mathcal{E}} A(e)$ . Now fix  $U$  and consider the natural inclusion map,  $U_T \hookrightarrow Z_T$ . Since the reduction of  $U_T$  is irreducible, it follows that  $\bar{U}_T$  maps to a point or onto an affine open of some irreducible component  $\Gamma_U$  of  $\bar{Z}_T$ . As  $U_T$  and  $Z_T$  are both connected to  $R(T)$ , the first case is not possible. Therefore, by Lemma 2.24 (ii),  $\bar{U}_T$  must inject into some such  $\Gamma_U$ . Let  $U'_T$  be the subaffinoid of  $Z_T$  which lies above the image of  $\bar{U}_T$ . By Lemma 2.24 (i), the inclusion of  $U_T$  into  $U'_T$  is surjective, and therefore an equality. As the  $U_T$  don't intersect, and  $U_T = U'_T$  for each  $U \in \mathcal{C}$ , the  $\Gamma_U$  must be distinct components.

Now, suppose  $e \in \mathcal{E}$ . By applying Lemma 2.25 to the inclusion of  $A(e)$  into  $Z_T$ , we see that  $A(e)$  must be contained in a residue class,  $R(y_e) := R_{Z_T}(y_e)$ , for some point  $y_e \in \bar{Z}_T(\mathbb{F}_K)$ . Thus there can be no irreducible components of  $\bar{Z}_T$  other than  $\{\Gamma_U : U \in \mathcal{C}\}$ . Moreover, it is clear that  $\bigcup_{e \in \mathcal{E}} A(e) = \bigcup_{e \in \mathcal{E}} R(y_e)$ . So using the fact that residue classes of an affinoid are connected<sup>12</sup>, it follows that the  $y_e$  are distinct and hence  $A(e) = R(y_e)$  for each  $e \in \mathcal{E}$ . From connectivity, we also have that  $y_e \in \Gamma_U \cap \Gamma_V$  whenever  $U \neq V$  and  $A(e) \subseteq U \cap V$ , and by Lemma 2.10 this must be a normal crossing. Therefore, as claimed, we have shown that  $\bar{Z}_T = \bar{\mathcal{C}} \setminus \bar{T}$ , and we use equality here to emphasize that the canonical reduction map on the  $Z_T(\mathbf{C})$  is compatible with the previously defined reduction map on

<sup>12</sup>If  $R$  is a residue class,  $A^\circ(R)$  is a local ring.

$C(\mathbf{C})$ .

To finish the proof, let  $T_1$  and  $T_2$  be two finite Galois stable sets of points of  $C(\bar{K})$  satisfying the above conditions on  $T$ , and such that  $\bar{T}_1 \cap \bar{T}_2 = \emptyset$ . Then  $Z := Z_{T_1} \cap Z_{T_2}$  is equal to  $Z_{T_1 \cup T_2}$ . Therefore,  $Z$  is a formal subdomain of both  $Z_{T_1}$  and  $Z_{T_2}$  by the compatibility of reduction maps. So  $C$  has semi-stable reduction  $\bar{Z}_{T_1} \cup \bar{Z}_{T_2} = \bar{C}$  with respect to the formal covering  $\{Z_{T_1}, Z_{T_2}\}$  (see [BL1, Def. 1.5]). Then, by precisely the same argument as used in the proof of Theorem 2.27,  $C$  has a semi-stable model over  $R_K$  whose reduction is isomorphic to  $\bar{C}$ .  $\square$

**Remark 2.37.** *As a consequence of Theorem 2.36 we have the result that whenever  $K$  is stable and satisfies Hypothesis B, every semi-stable curve over  $R_K$  can be constructed by gluing together wide opens taken out of curves with **good reduction** over  $R_K$ . Crossings of distinct irreducible components are created by gluing two annuli at the ends of two distinct wide opens, while self-intersections within a component are created by gluing two annuli at distinct ends of a single wide open.*

**Lemma 2.38.** *Suppose  $D$  is a closed disk and  $U$  is either an open disk or open annulus in a smooth complete curve  $C$ , all defined over  $K$ , such that  $D \cap U \neq \emptyset$ . Then either  $D \subseteq U$ ,  $U \subseteq D$ ,  $D \cup U$  is an open disk, or  $D \cup U = C \cong \mathbf{P}^1$  and  $U$  is an open disk.*

*Proof.* We can assume  $K = \mathbf{C}$  and  $g(C) > 0$ . When  $U$  is an open disk the lemma follows from Proposition 5.4(a) of [BL1]. So suppose  $U$  is an open annulus,  $U \not\subseteq D$  and  $D \not\subseteq U$ . We first show that every concentric circle  $R$  of  $U$  which intersects  $D$  must be contained in  $D$ . Indeed, applying Proposition 5.4 (c) to the height 1 annulus  $R$  and the disk  $D$ , and using the fact that  $D \not\subseteq R$ , we can conclude that  $R$  is contained in some closed disk  $E$ . Then by Proposition 5.4(a) of [BL1], we have  $D \subseteq E$  or  $E \subseteq D$ . Either way, it follows that  $R \subseteq D$ .

Now choose a parameterization,  $\psi : A_K(r, s) \xrightarrow{\cong} U$ . By the preceding argument, we can then choose  $t \in \mathcal{R}(r, s)$  such that  $Y_t := \psi(C_K[t]) \subseteq D$ . Then  $C \setminus Y_t$  and  $U \setminus Y_t$  have two connected components (each). Since  $U$  is connected,  $U \setminus Y_t \not\subseteq C \setminus D$ . Thus  $\exists u \in \mathcal{R}(r, s)$  such that  $u \neq t$  and  $Y_u \subseteq D$ . We can assume that  $u < t$  and  $Y_u$  is contained in the connected component  $Z$  of  $C \setminus Y_t$  which lies inside  $D$ . Because  $A_K[u, t]$  is connected, it follows that  $\psi(A_K[u, t]) \subseteq Z$ .

Now choose a  $P \in Z \setminus U$ , and let  $\phi : B_K[1] \xrightarrow{\cong} D$  be any parameterization such that  $\phi(0) = P$ . We may assume that  $\phi(C_K[v]) = Y_v$  whenever  $Y_v \subseteq D$ . Thus,  $\phi(A_K(r, 1]) = U \cap D$  and  $s > 1$ . Finally, we let  $V = D \cup U$ . Then  $V$  is a wide open with one end and  $\phi^{-1}B_K[t]$  is an underlying affinoid for  $t \in \mathcal{R}(r, 1]$ . Hence  $g(V) = 0$ , and so by the Riemann Existence Theorem,  $V$  is isomorphic to  $\mathbf{P}^1$  minus a closed disk (in particular, an open disk).  $\square$

If  $\mathcal{C}$  is a semi-stable covering of  $C$ , we define a **residue class** of  $\mathcal{C}$  to be either a residue class of  $U^u$  or a component of  $U \setminus U^u$ , for some  $U \in \mathcal{C}$ .

**Corollary 2.39.** *Suppose  $\mathcal{C}$  is a stable covering of  $C$ . Then every closed disk  $D$  in  $C$  is contained in a residue class of  $\mathcal{C}$ .*

*Proof.* Extend scalars to  $\mathbf{C}$ . The curve  $C$  is not isomorphic to  $\mathbf{P}^1$ , as  $\mathbf{P}^1$  does not have a stable covering. Let  $R$  be a residue class of  $\mathcal{C}$  such that  $D \cap R \neq \emptyset$ , and suppose  $D \not\subseteq R$ . First suppose  $R$  is an open disk. If necessary, refine  $\mathcal{C}$  to a semi-stable covering  $\mathcal{C}'$  for which all underlying affinoids have smooth reduction, none are closed disks, and  $R$  is a residue class of  $U^u$  for some  $U \in \mathcal{C}'$ . By the previous lemma, we have  $R \subseteq D$ .

This latter containment implies that  $U^u \cap D$  is a non-empty affinoid with good reduction. Every such affinoid is a Zariski subaffinoid of a closed disk  $E_1$  in  $D$ , because  $D$  is a closed disk. Since  $U^u$  has good reduction,  $E_1$  is a disk, and  $C \not\cong \mathbf{P}^1$ , it follows that  $U^u$  is a Zariski subaffinoid of  $E_1$ .

Set  $U_1 = U$ . Then  $U_1^u$  is not equal to  $E_1$ , since none of the underlying affinoids in  $\mathcal{C}'$  are disks. Therefore, there exists a residue disk  $R_1$  of  $E_1$  such that  $A_1 := R_1 \cap U_1$  is a component of  $U_1 \setminus U_1^u$  (an open annulus). Now  $E_2 := R_1 \setminus A_1$  is a closed disk. Let  $U_2$  be the other element of  $\mathcal{C}'$  containing  $A_1$ . By the same argument as above, and the fact that both  $U_2^u$  and  $E_2$  are connected to  $A_1$ , it follows that  $U_2^u$  must be a Zariski subaffinoid of  $E_2$ . Again (for  $i = 2$  now), we must have  $U_i^u \neq E_i$ . Proceeding in this manner, we eventually exhaust the underlying affinoids of  $\mathcal{C}'$  or find a  $V \in \mathcal{C}'$  such that  $V^u$  is a closed disk. Thus, we have a contradiction.

Now suppose  $R$  is an annulus. If an annulus at one end of  $R$  is contained in  $D$ , and  $U^u$  is connected to  $R$  at that end, for some  $U \in \mathcal{C}$ , then  $D$  intersects every residue class of  $U^u$ . In particular, it intersects an open disk. Now apply the above argument. □

**Theorem 2.40.** *Suppose  $C$  is a smooth complete curve over a stable field  $K$  satisfying Hypothesis B, and  $\mathcal{D}$  is a finite (possibly empty) collection of disjoint closed disks in  $C$  all defined over  $K$ . Then there exists a semi-stable covering  $\mathcal{C}$  of  $C$  over some finite extension of  $K$  such that:*

- (1) for each  $D \in \mathcal{D}$ ,  $D = U_D^u$  for some  $U_D \in \mathcal{C}$ , and
- (2)  $\mathcal{C} \setminus \{(U_D, D) : D \in \mathcal{D}\}$  is a semi-stable covering of  $W := C \setminus \bigcup_{D \in \mathcal{D}} D$ .

*Proof.* If  $\mathcal{D}$  is empty, or if  $|\mathcal{D}| = 1$  and  $g(C) = 0$ , the theorem follows directly from Theorem 2.36 (and Deligne-Mumford – van der Put).

Otherwise, suppose we have a semi-stable covering  $\mathcal{C}$  of  $C$  which is *compatible* with  $\mathcal{D}$ , in the sense that each  $D \in \mathcal{D}$  is either contained in a residue class of  $\mathcal{C}$  or equal to  $U^u$  for some  $U \in \mathcal{C}$ . Then we can refine  $\mathcal{C}$  to obtain a covering that satisfies the conclusions of the theorem. Indeed, suppose  $D \in \mathcal{D}$  and  $D \neq U^u$  for any  $U \in \mathcal{C}$ . Then  $D$  is contained in a residue class  $R$  of  $\mathcal{C}$ , and there are three possibilities to consider. First,  $D$  could be contained in a residue disk  $R$  of  $U^u$  for some  $U \in \mathcal{C}$ . In this case we refine our covering to

$$\mathcal{C}_D := \mathcal{C} \setminus \{(U, U^u)\} \cup \{(U \setminus D, U^u \setminus R), (R, D)\}.$$

The second possibility is that  $D$  is contained in a residue annulus  $R$  of some  $U^u$ . Applying Lemma 2.38 from above, there must then be a concentric circle

$T$  in  $R$ , and a residue disk  $S$  in  $T$ , such that  $D \subseteq S$ . In this case, we let

$$\mathcal{C}_D := \mathcal{C} \setminus \{(U, U^u)\} \cup \{(U \setminus T, U^u \setminus R), (R \setminus D, T \setminus S), (S, D)\}.$$

Finally, the residue class  $R$  which contains  $D$  may be a connected component of  $U \cap V$  for two distinct  $U, V \in \mathcal{C}$ . Again there must be a concentric circle  $T$  in  $R$  and a residue disk  $S$  in  $T$  so that  $D \subseteq S$ . Let  $R_U$  and  $R_V$  be the connected components of  $R \setminus T$  which are connected to  $U$  and  $V$  respectively. Let  $\hat{U} = (U \setminus R) \cup R_U$  and  $\hat{V} = (V \setminus R) \cup R_V$ . Then we may take as our refined cover,

$$\mathcal{C}_D := \mathcal{C} \setminus \{(U, U^u), (V, V^u)\} \cup \{(\hat{U}, U^u), (\hat{V}, V^u), (R \setminus D, T \setminus S), (S, D)\}.$$

After applying this procedure finitely many times, we are done.

The only issue remaining is that of finding a compatible covering as a starting point. If  $\mathcal{C}$  is a stable covering then it is compatible with  $\mathcal{D}$  by Corollary 2.39. So when  $g \geq 2$  we are done by Theorem 2.36. If  $g = 0$ , and  $D_1, D_2 \in \mathcal{D}$  with  $D_1 \neq D_2$ , then  $\mathcal{C} := \{(C \setminus D_1, D_2), (C \setminus D_2, D_1)\}$  is compatible with  $\mathcal{D}$ . If  $g(C) = 1$ ,  $D \in \mathcal{D}$ , and  $U$  is the largest open disk in  $C$  containing  $D$ , then  $\mathcal{C} := \{(C \setminus D, C \setminus D), (U, D)\}$  is compatible with  $\mathcal{D}$ . So in each case we are able to construct the desired covering of  $C$ .  $\square$

**Remark 2.41.** *If  $g(C) \geq 2$ , or  $g(C) = 0$  and  $|\mathcal{D}| \geq 3$ , or  $g(C) = 1$  and  $|\mathcal{D}| \geq 1$ , there exists a final object  $\mathcal{C}_{\mathcal{D}}$  in the category of such coverings. In these cases,  $C \setminus \bigcup_{D \in \mathcal{D}} U_D$  is the minimal underlying affinoid of  $W := C \setminus \bigcup_{D \in \mathcal{D}} D$ .*

**Corollary 2.42.** *Let  $f$  be a meromorphic function with finitely many zeroes and poles on a wide open  $W$  over a stable field  $K$  satisfying Hypothesis B. Then there is a semi-stable covering  $\mathcal{C}$  of  $W$  over a finite extension of  $K$  such that for each  $U \in \mathcal{C}$ ,  $U^u$  has good reduction and all the zeroes and poles of  $f$  are contained in  $\bigcup_{U \in \mathcal{C}} U^u$ .*

*Proof.* Glue in disks to get a complete curve  $C$ . Let  $\mathcal{D}$  be the union of  $CC(C \setminus W)$  with a finite collection of disjoint closed disks in  $W$  which contain the support of  $f$ . Apply the theorem to get a semi-stable covering  $\mathcal{C}_1$  of  $C$  over some finite extension of  $K$ , and then throw out those  $U \in \mathcal{C}_1$  for which  $U^u \in CC(C \setminus W)$ . This yields a semi-stable covering  $\mathcal{C}_2$  of  $W$  such that all the zeroes and poles of  $f$  are contained in  $\bigcup_{U \in \mathcal{C}_2} U^u$ . Let  $\mathcal{S}$  be the collection of singular residue classes in  $U^u$  for all  $U \in \mathcal{C}_2$ . For each  $R \in \mathcal{S}$ , choose a concentric circle  $A_R \subset R$  (such an  $R$  is an open annulus). Then

$$\mathcal{C} := \{(U \setminus \bigcup_{R \in \mathcal{S}} A_R, U^u \setminus \bigcup_{R \in \mathcal{S}} R) : U \in \mathcal{C}_2\} \cup \{(R, A_R) : R \in \mathcal{S}\}$$

satisfies the requirements of the corollary.  $\square$

Our final result of this section is a lemma which will play a key role in the proof of our main theorem (Theorem 9.2).

**Lemma 2.43.** *Suppose  $W$  is a connected wide open over a stable field  $K$  satisfying Hypothesis B, with minimal underlying affinoid  $W^u$ , and  $X \subset W$  is an affinoid subdomain with smooth irreducible (connected) reduction such that  $g(W) = g(\overline{X}^c) > 0$ . If  $X$  is connected to all but at most one component of  $W \setminus W^u$ , then  $W$  is a basic wide open and  $X$  is a Zariski subaffinoid of  $W^u$ .*

*Proof.* First glue disks to  $W$  to obtain a smooth connected complete curve  $C$  over  $K$ . Then by Theorem A1<sup>13</sup> of [C3], there exists a semi-stable model  $\mathcal{T}$  of  $C$  over a finite extension  $E$  of  $K$ , and a subset  $S$  of the set of components of  $\overline{\mathcal{T}}$  such that  $X_E = X(\mathcal{T}, S)$ . Moreover, there exists an  $s \in S$  such that  $X(\mathcal{T}, s)$  is a Zariski subaffinoid of  $X_E$ .

Let  $\mathcal{C} := \mathcal{C}_{\mathcal{T}}$  be the semi-stable covering of  $C_E$  associated to  $\mathcal{T}$  by Theorem 2.36. This implies by Proposition 2.34 that  $Betti(\Gamma_{\mathcal{C}}) = 0$  and  $g(z) = 0$  for all  $z \in S$  different from  $s$ . It follows that  $C_E$  has good reduction isomorphic to  $\overline{X}_E^c$ ,  $X_E$  is a Zariski subaffinoid of  $C_E$ , and each affinoid disk in  $C \setminus W$  is contained in a residue class of  $C_E$ . Furthermore, the statement that  $X$  is connected to all but at most one end of  $W$  implies that the elements of  $C \setminus W$  lie in distinct residue classes of  $C_E$ , and that the complement of these residue classes is the minimal underlying affinoid of  $W_E$  which equals  $W_E^u$ .

We now know that  $A^\circ(W_E^u) \cong A(W_E^u) \cap A^\circ(X_E)$ , under restriction  $\rho$ , and that  $A(W_E^u) \cong A(W^u) \otimes_K E$ . Also,  $A^\circ(W^u) = A^\circ(W_E^u) \cap A(W^u)$  and  $A^\circ(X_E) = A^\circ(X) \otimes_{R_K} R_E$  because  $X$  has good reduction.

It follows that there is some non-zero element  $m \in R_K$ ,  $|m| < 1$ , such that  $mA^\circ(W_E^u) \subset A^\circ(W^u) \otimes_{R_K} R_E$ . So if  $c \in A^\circ(W_E^u)$ ,  $\rho(c) = \sum_i r_i b_i$ , where  $r_1, \dots, r_n$  is a basis for  $R_E$  over  $R_K$  and  $b_i \in A^\circ(X)$ . It follows that  $mb_i = \rho(a_i)$  for some  $a_i \in \rho(A^\circ(W^u))$ . Thus  $a_i/m \in A^\circ(W_E^u)$ , and so  $a_i/m \in A^\circ(W^u)$ . Therefore  $A^\circ(W_E^u) = A^\circ(W^u) \otimes_{R_K} R_E$ . This implies that  $W^u$  has good reduction, which completes the proof.  $\square$

## 2.4 Riemann-Hurwitz for Wide Opens

For this entire section we assume that  $K$  is a stable field satisfying Hypothesis B. Let  $\mathcal{A}$  be an *oriented* annulus over  $K$ . Suppose  $f$  is a function on  $\mathcal{A}$ , and  $\omega$  a differential, each with no zeroes or poles in  $\mathcal{A}(\mathbf{C})$ . Then we define  $\text{ord}_{\mathcal{A}} f = \text{res}_{\mathcal{A}}(df/f)$ , which is an integer (see the proof of [C1, Lemma 2.1]), and  $\text{ord}_{\mathcal{A}} \omega = \text{ord}_{\mathcal{A}}(\omega/dz)$ , for any  $z \in A(\mathcal{A})^*$  with  $\text{ord}_{\mathcal{A}} z = 1$  (which is independent of the choice of  $z$ ). Using this definition, we can also define  $\text{ord}_e$  at any end  $e$  of a wide open  $W/K$ . Indeed, suppose  $\nu$  is either a meromorphic function or differential on  $W$ , with finitely many zeroes and poles in  $W(\mathbf{C})$ . Over some finite extension  $L$  of  $K$ ,  $W_L$  will have an underlying affinoid  $X_L$  containing the support of  $\nu$ . Let  $\mathcal{A}$  be the component of  $W_L \setminus X_L$  corresponding to a fixed  $e \in \mathcal{E}(W)$ , and let  $\psi: A_K(r, s) \rightarrow \mathcal{A}$  be an isomorphism such that  $\psi(A_K(t, s))$  is connected to  $X$  whenever  $r < t < s$ . Then we define the **inherited orientation** on  $\mathcal{A}$  by  $\text{res}_{\mathcal{A}} = \text{res}_{r,s} \circ \psi^*$ , and we set  $\text{ord}_e \nu = \text{ord}_{\mathcal{A}} \nu$ .

<sup>13</sup>The proof of this result was based on Proposition 2.1 of [C2], which is now a special case of Theorem 2.36.

Let  $\text{Div}(W) := \mathbb{Z}^{W(\mathbf{C}) \cup \mathcal{E}(W)}$ , and for any  $D \in \text{Div}(W)$  let

$$\deg D = \sum_{P \in W(\mathbf{C})} D(P) + \sum_{e \in \mathcal{E}(W)} D(e).$$

Then for  $\nu$  as above, set  $(\nu) = (\nu)_{fin} + (\nu)_{inf}$  where

$$(\nu)_{fin} = \sum_{P \in W(\mathbf{C})} \text{ord}_P \nu \quad \text{and} \quad (\nu)_{inf} = \sum_{e \in \mathcal{E}(W)} \text{ord}_e \nu.$$

**Lemma 2.44.** *Suppose  $f$  is a meromorphic function and  $\omega$  a meromorphic differential on  $B(1) := B_{\mathbf{C}}(1)$ , each with finitely many zeroes and poles, and each supported on  $B[r] := B_{\mathbf{C}}[r]$  for some  $r < 1$ . Let  $\mathcal{A} = A_{\mathbf{C}}(r, 1)$ , oriented so that  $\text{res}_{\mathcal{A}} = \text{res}_{r,1}$  (so not the inherited orientation from  $B(1)$  as a wide open). Then*

$$\text{ord}_{\mathcal{A}} f = \sum_{P \in B(1)} \text{ord}_P f, \quad \text{ord}_{\mathcal{A}} \omega = \sum_{P \in B(1)} \text{ord}_P \omega, \quad \text{and} \quad \text{res}_{\mathcal{A}} \omega = \sum_{P \in B(1)} \text{res}_P \omega.$$

*Proof.* Let  $z$  be the natural parameter on  $B(1)$ . For the first equation, suppose  $f$  is supported on  $\{P_1, \dots, P_n\}$  with  $\text{ord}_{P_i} f = e_i$  and  $z(P_i) = \alpha_i$ . By Weierstrass Preparation Theorem, we may write  $f(z) = \prod_{i=1}^n (z - \alpha_i)^{e_i} \cdot u(z)$  where  $u$  is a unit. Then

$$\text{ord}_{\mathcal{A}} f = \sum_{i=1}^n \text{res}_{\mathcal{A}} \left( \frac{e_i dz}{z - \alpha_i} \right) = \sum_{i=1}^n e_i = \sum_{P \in B(1)} \text{ord}_P f.$$

The other two equations follow from essentially the same argument.  $\square$

**Theorem 2.45.** *Let  $f$  be a rigid function and  $\omega$  a differential on  $W$ , each with finitely many poles and zeroes in  $W(\mathbf{C})$ . Then (i)  $\deg(f) = 0$ , (ii)  $\deg(\omega) = 2g(W) - 2$ , and*

$$\sum_{P \in W(\mathbf{C})} \text{res}_P \omega + \sum_{e \in \mathcal{E}(W)} \text{res}_e \omega = 0. \quad (iii)$$

*Proof.* Attach disks at the ends of  $W$  to obtain a smooth projective curve  $C$ . For any rational function  $g$  on  $C$ , it follows immediately from Lemma 2.44 that  $\deg(g|_W) = 0$ .

For more general  $f$ , suppose first that  $(W, X)$  is a basic wide open pair,  $X$  has good reduction and  $(f)_{fin}$  is supported on  $X$ . In this case, there exists a  $g \in \mathcal{O}_C$  and a Zariski subaffinoid  $Y$  of  $X$  such that  $f/g$  is regular on  $Y$  and  $|(f/g) - 1|_Y < 1$  (in particular, we could choose  $Y$  so that  $f$  and  $g$  have no poles or zeroes on  $Y$ ). It follows that there is a wide open  $V$ , with  $Y \subset V \subseteq W$ , such that  $(V, Y)$  is a basic wide open pair and  $|(f/g) - 1|_V < 1$ . Hence,  $(f|_V) = (g|_V)$ . Now, we have a natural map  $\beta: \text{Div}(W) \rightarrow \text{Div}(V)$ . Indeed, the elements of  $\mathcal{E}(V)$  are in one-to-one correspondence with the connected components of  $V \setminus Y$ , which in turn are in one-to-one correspondence (by intersection) with

the connected components of  $W \setminus Y = (W \setminus X) \cup (X \setminus Y)$ . Thus, as  $e \in \mathcal{E}(W)$  corresponds to a unique component of  $W \setminus X$ , it then corresponds to a unique end of  $V$  which we take to be  $\beta(e)$ . Similarly, if  $P \in W(\mathbf{C})$ , we let  $\beta(P) = P$  if  $P \in V(\mathbf{C})$ , and the element of  $\mathcal{E}(V)$  corresponding to the component of  $X \setminus Y$  which contains  $P$  otherwise. Extend this map by linearity. Since  $\deg \beta(D) = \deg D$  and  $(f|_V) = \beta(f)$ , we have

$$\deg(f) = \deg(f|_V) = \deg(g|_V) = 0.$$

To complete the proof that  $\deg(f) = 0$ , let  $\mathcal{C}$  be a semi-stable covering of  $W$  such that  $U^u$  has good reduction and  $(f|_U)_{fin}$  is supported on  $U^u$  for each  $U \in \mathcal{C}$  (which exists by Corollary 2.42). Then

$$(f) = \sum_{U \in \mathcal{C}} (f|_U),$$

where we regard both sides as elements of

$$\{D \in \mathbb{Z}^{W(\mathbf{C}) \cup \bigcup_{U \in \mathcal{C}} \mathcal{E}(U)} : D(a) = -D(b) \text{ if } a \in \mathcal{E}(U), b \in \mathcal{E}(V), U \neq V, U_a = V_b\}.$$

Therefore,

$$\deg(f) = \sum_{U \in \mathcal{C}} \deg(f|_U) = 0.$$

Statements (ii) and (iii) are clearly true whenever  $\omega = \eta|_W$  for  $\eta \in \Omega_C^1$ , by Lemma 2.44. Moreover, the general case of (ii) then follows from (i) and the fact that  $(f\omega) = (f) + (\omega)$ . Finally, the general case of (iii) will follow once we know it for basic wide opens by an argument similar to that above.

So suppose  $(W, X)$  is a basic wide open pair, with  $X$  and  $C$  as above. For a reduced affinoid  $X$  over  $K$  and  $\omega \in \Omega_{X/K}^1$ , we set

$$|\omega|_X = \inf\{|a| : a \in K, \omega \in aA^o(X)dA^o(X)\}.$$

Using Riemann-Roch, we can find  $\eta \in \Omega_C^1$  such that  $\omega - \eta$  has no poles on  $X$  and  $|\omega - \eta|_X < \epsilon$ . Note that  $\omega - \eta$  extends to a regular differential on a wide open neighborhood  $V$  of  $X$  in  $W$ . Then statement (iii) for  $W$  follows from the general fact that if  $(V, X)$  is a basic wide open pair,  $\omega \in \Omega_{V/K}^1$ ,  $|\omega|_X < \epsilon$  and  $e \in \mathcal{E}(V)$  then  $|\text{res}_e \omega| < \epsilon$ . Indeed, let  $T: U \rightarrow A(1, \infty)$  be a parameter on the component  $U$  of  $V \setminus X$  corresponding to  $e$ , such that  $|T(x)| \rightarrow 1$  as  $x \rightarrow X$ . Suppose on  $U$

$$\omega = \sum_{n=-\infty}^{\infty} a_n T^n dT.$$

Then  $|\omega|_X = \max\{|a_n| : -\infty < n < \infty\}$ . So  $|\text{res}_e \omega| = |a_{-1}| < \epsilon$ . □

Suppose  $f: W \rightarrow V$  is a finite map. As  $f$  is finite,  $f$  naturally maps  $\mathcal{E}(W)$  to  $\mathcal{E}(V)$ . For  $a \in W(\mathbf{C}) \cup \mathcal{E}(W)$ , let  $\delta_f(a) = \text{ord}_a f^* dT$  where  $T$  is a parameter

at  $b := f(a)$  such that  $\text{ord}_b T = 1$ . When  $a$  and  $b$  are ends, there exist annuli  $\mathcal{A}$  and  $\mathcal{B}$  at  $a$  and  $b$  such that  $f$  restricts to a finite étale map from  $\mathcal{A}$  onto  $\mathcal{B}$ . Let  $e_f(a)$  be the degree of this map. Otherwise, at a point in  $W(\mathbf{C})$ , let  $e_f(a)$  denote the usual ramification index.

**Lemma 2.46.** *With notation as above, if  $\omega$  is a differential with finitely many zeroes and poles on  $W$ , then*

$$\text{ord}_a f^* \omega = e_f(a) \text{ord}_b \omega + \delta_f(a).$$

*Proof.* First suppose  $a \in \mathcal{E}(W)$ , and let  $\mathcal{A}$  and  $\mathcal{B}$  be annuli at  $a$  and  $b$  such that  $\omega$  is regular and non-vanishing on  $\mathcal{B}$ . Choose parameters  $S$  and  $T$  on  $\mathcal{A}$  and  $\mathcal{B}$  respectively, such that  $\text{ord}_a S = \text{ord}_b T = 1$ . Then  $f^*T|_{\mathcal{A}} = S^e g(S)$  and  $\omega|_{\mathcal{B}} = T^d h(T) dT$ , where  $g$  is a unit on  $\mathcal{A}$  with  $\text{ord}_a g = 0$ ,  $h$  is unit on  $\mathcal{B}$  with  $\text{ord}_b h = 0$ ,  $e = e_f(a)$ , and  $d = \text{ord}_b \omega$ . So

$$(f^* \omega)|_{\mathcal{A}} = (S^e g(S))^d h(S^e g(S)) f^* dT,$$

from which the lemma follows.

The proof when  $a \in W(\mathbf{C})$  is very similar. Let  $\mathcal{A}$  and  $\mathcal{B}$  be the stalks at  $a$  and  $b$ . Then after choosing uniformizers  $S$  and  $T$ , respectively, the map  $f : \mathcal{A} \rightarrow \mathcal{B}$  is given by a homomorphism between formal power series rings over  $\mathbf{C}$ . Thus, we have  $\omega = T^d h(T) dT$  and  $f^*T = S^e g(S)$ , where  $e = e_f(a)$ ,  $d = \text{ord}_b \omega$ , and  $g$  and  $h$  are formal power series with nonzero constant terms. The lemma follows from the above computation and the fact that  $h(S^e g(S))$  will again have nonzero constant term.  $\square$

**Corollary 2.47.** *Suppose that  $|e_f(a)| = 1$  in  $K$ , or that  $e_f(a) \neq 0$  in  $K$  and  $a \in W(\mathbf{C})$ . Then  $\delta_f(a) = e_f(a) - 1$ .*

*Proof.* Keeping the same notation as above, we compute  $\delta_f(a)$  directly from the definition.

$$\delta_f(a) = \text{ord}_a dT = \text{ord}_a d(S^e g(S)) = \text{ord}_a (eg(S) + Sg'(S)) + e - 1$$

When  $a \in W(\mathbf{C})$  and  $e_f(a) \neq 0$  in  $K$ , this equals  $e - 1$ , since the power series  $eg(S) + Sg'(S)$  must have nonzero constant term. On the other hand, if  $a \in \mathcal{E}(W)$  and  $|e_f(a)| = 1$ , it is straightforward to show that  $eg(S) + Sg'(S)$  has constant absolute value on  $\mathcal{A}$ . So either way we are done.  $\square$

**Theorem 2.48.** *Suppose  $f : W \rightarrow V$  is a finite map of wide opens of degree  $d$ . Then*

$$2g(W) - 2 = d(2g(V) - 2) + \sum_{a \in W(\mathbf{C}) \cup \mathcal{E}(W)} \delta_f(a).$$

*Furthermore, under the hypotheses of Corollary 2.47, this is*

$$d(2g(V) - 2) + \sum_{a \in W(\mathbf{C}) \cup \mathcal{E}(W)} (e_f(a) - 1).$$

*Proof.* Let  $\omega$  be a non-zero meromorphic differential on  $V$ . Then the degree of  $f^*\omega$  must be  $2g(W) - 2$  by Theorem 2.45. On the other hand, we obtain the right hand side of the equation if we compute  $\deg(f^*\omega)$  using Lemma 2.46, Corollary 2.47, and the fact that

$$\sum_{\substack{x \in W(\mathbf{C}) \\ f(x)=y}} e_f(x) = \sum_{\substack{a \in \mathcal{E}(W) \\ f(a)=b}} e_f(a) = d.$$

□

**Proposition 2.49.** *Suppose  $(W, W^u)$  and  $(V, V^u)$  are basic wide open pairs, and  $f: W \rightarrow V$  is a finite map such that  $f(W^u) = V^u$ . Let  $X$  and  $Y$  be the completions of  $\overline{W^u}$  and  $\overline{V^u}$ , and let  $\bar{f}: X \rightarrow Y$  be the induced map. If  $\bar{f}$  is separable, then*

$$\delta_f(a) = \text{length}(\Omega_{X/Y})_a.$$

*Proof.* First, we can lift  $\bar{f}$  to a map,  $g: C_X \rightarrow C_Y$ , between complete liftings of  $X$  and  $Y$ . There must exist wide open neighborhoods  $W' \subseteq W$  and  $V' \subseteq V$  of  $W^u$  and  $V^u$ , and embeddings  $\phi_X: W' \rightarrow C_X$  and  $\phi_Y: V' \rightarrow C_Y$ , such that  $f(W') = V'$ ,  $\phi_X|_{W^u}$  and  $\phi_Y|_{V^u}$  are the natural inclusions, and

$$\overline{g \circ \phi_X|_{W^u}} = \overline{\phi_Y \circ f|_{W^u}}.$$

Now, Hartshorne's version of Hurwitz's Theorem (see [H, Cor. 2.4]) implies the proposition for

$$h := \phi_Y^{-1} \circ g \circ \phi_X: W' \rightarrow V'.$$

The proposition follows because  $\delta_f(a) = \delta_f(a') = \delta_g(a'')$ , where  $a'$  is the component of  $W' \setminus W^u$  corresponding to  $a$  and  $a'' = \phi_X(a')$ . □

### 3 $X_0(p^n)$ and its Subspaces

Now that the rigid analytic foundation has been laid, we turn our focus specifically on the curve  $X_0(p^n)$ , which we will always think of in moduli-theoretic terms. More precisely, we think of  $X_0(p^n)$  as the rigid analytic curve over  $\mathbb{Q}_p$  whose points over  $\mathbb{C}_p$  are in a one-to-one correspondence with (isomorphism classes of) pairs,  $(E, C)$ , where  $E/\mathbb{C}_p$  is a generalized elliptic curve and  $C$  is a cyclic subgroup of order  $p^n$ . We implicitly make use of this correspondence when we speak loosely of “the point  $(E, C)$ .” There are various natural maps from  $X_0(p^n)$  to  $X_0(p^m)$  which can be defined by way of this moduli-theoretic interpretation of points, and we begin this section by fixing notation for these fundamental maps.

**Definition 3.1.** *First let*

$$\pi_f, \pi_\nu: \prod_{n \geq 1} X_0(p^n) \rightarrow \prod_{n \geq 0} X_0(p^n)$$

be the maps given by  $\pi_f(E, C) = (E, pC)$  and  $\pi_\nu(E, C) = (E/C[p], C/C[p])$ , where  $C[p^i]$  is the kernel of multiplication by  $p^i$  in  $C$ . Then by letting  $\pi_{ab} = \pi_f^b \circ \pi_\nu^a$  we get maps

$$\pi_{ab} : \prod_{n \geq a+b} X_0(p^n) \rightarrow \prod_{n \geq 0} X_0(p^n).$$

**Remark 3.2.** This definition is identical to the definition in [C3, §1]. We also note that over  $\mathbb{C}$ ,  $\pi_{ab}$  corresponds to the map on the upper half plane which takes  $z$  to  $p^a z$ .

Another map crucial to this paper is the Atkin-Lehner involution,

$$w : \prod_{n \geq 0} X_0(p^n) \rightarrow \prod_{n \geq 0} X_0(p^n),$$

which is defined by the formula,

$$w_n(E, C) = (E/C, E[p^n]/C),$$

where  $w_n := w|_{X_0(p^n)}$ . The Atkin-Lehner involution is compatible with the level-lowering maps in the sense that  $\pi_f \circ w = w \circ \pi_\nu$  (or equivalently,  $w \circ \pi_f = \pi_\nu \circ w$ , since  $w$  is an involution).

### 3.1 Canonical Subgroups and Supersingular Annuli

In this section we introduce some natural rigid subspaces of  $X_0(p^n)$  over finite extensions of  $\mathbb{Q}_p$  using the theory of the canonical subgroup, which we now review and extend, citing [Bu, §3] as our primary reference.<sup>14</sup> If  $E$  is an elliptic curve over  $\mathbb{C}_p$ , we let  $h(E)$  denote the minimum of 1 and the valuation of a lifting of the Hasse invariant of the reduction of a non-singular model of  $E \bmod p$ , if it exists, and 0 otherwise<sup>15</sup> (this is denoted by  $v(E)$  in [Bu]). In [Ka, §3], Katz constructed a rigid analytic section  $s_1$  of  $\pi_f : X_0(p) \rightarrow X(1)$  over the wide open  $W_1$  whose  $\mathbb{C}_p$ -valued points are represented by generalized elliptic curves  $E$  such that  $h(E) < p/(p+1)$ , when  $p \geq 5$ . Both  $W_1$  and  $s_1$  are defined over  $\mathbb{Q}_p$ . Changing notation slightly from [Bu], we let  $K_1(E) \subseteq E$  denote the subgroup of order  $p$  for which  $s_1(E) = (E, K_1(E))$ , and we call  $K_1(E)$  the canonical subgroup of order  $p$ .

Using Theorem 3.3 of [Bu], we can also define canonical subgroups of higher order. For  $n \geq 1$  we generalize  $W_1$  by taking  $W_n$  to be the wide open in  $X(1)$  where  $h(E) < p^{2-n}/(p+1)$  (the complement of finitely many affinoid disks, one in each supersingular residue class). For  $E \in W_n$  we then define  $K_n(E)$  inductively, as in [Bu, Def 3.4], as the preimage of  $K_{n-1}(E/K_1(E))$  under the natural projection from  $E \rightarrow E/K_1(E)$ . This is a cyclic subgroup

<sup>14</sup>Although Buzzard works over a complete DVR, all of his results can be extended to complete local rings.

<sup>15</sup>As pointed out in [BL1, Rmk. 6.4], the good reduction of  $E$  is well defined if it exists.

when  $E \in W_n$  by [Bu, Thm 3.3], and we call it the canonical subgroup of order  $p^n$ .<sup>16</sup> Thus, when  $E$  has supersingular reduction, either  $h(E) \geq p/(p+1)$  (and  $E$  is too-supersingular in the language of [Bu]), or there is a largest  $n \geq 1$  for which  $K_n(E)$  can be defined. In the first case, we define the **canonical subgroup** of  $E$ , denoted  $K(E)$ , to be the trivial subgroup, and in the second we let  $K(E) = K_n(E)$  for this largest  $n$ . Whenever  $E/\mathbb{C}_p$  has ordinary reduction (by this we mean ordinary good or multiplicative<sup>17</sup>) we let  $K(E)$  be the  $p$ -power torsion of  $E$  which is contained in the kernel of reduction, which does not depend on the good or multiplicative model.

It is important to note that  $s_1$  also generalizes, in the sense that the map defined by  $s_n(E) = (E, K_n(E))$  is also a rigid analytic section of  $\pi_{0n} : X_0(p^n) \rightarrow X(1)$  over the wide open  $W_n$ . To see this, first regard  $X_0(p^n)$ , for  $n > 1$ , as the normalization of the fiber product of  $X_0(p^{n-1})$  with itself over  $X_0(p^{n-2})$  via the maps  $\pi_f$  and  $\pi_\nu$ . More specifically, let

$$\psi_n : X_0(p^n) \rightarrow X_0(p^{n-1}) \times_{\pi_f, \pi_\nu} X_0(p^{n-1})$$

be the isomorphism described by  $\psi_n = (\pi_\nu, \pi_f)$  (after normalization of the right hand side). Now assume that  $s_{n-1}$  is rigid analytic. Applying Theorem 3.3 of [Bu], it is straightforward to verify that over  $W_n$  we have

$$\pi_f \circ s_{n-1} \circ \pi_{1n-2} \circ s_{n-1} = \pi_\nu \circ s_{n-1}.$$

Thus we may define a rigid analytic map from  $W_n$  to  $X_0(p^n)$  by

$$s_n := \psi_n^{-1} \circ (s_{n-1} \circ \pi_{1n-2} \circ s_{n-1}, s_{n-1}).$$

Using Theorem 3.3 of [Bu] again, we see that this map does indeed take  $E$  to  $(E, K_n(E))$ . So by induction we are done. Note that both  $W_n$  and  $s_n$  are defined over  $\mathbb{Q}_p$ .

Another way to focus on rigid subspaces of  $X_0(p^n)$  is to fix the isomorphism class of the reduction of  $E$ . In particular, we make the following definition.

**Definition 3.3.** *For a fixed elliptic curve  $A$  over a finite field  $\mathbb{F}$ , let  $W_A(p^n)$  represent the rigid subspace of  $X_0(p^n)$  (over  $\mathbb{Q}_p \otimes W(\mathbb{F})$ ) whose points over  $\mathbb{C}_p$  are represented by pairs  $(E, C)$  with  $\bar{E} \cong A$ .*

Of course,  $W_A(1)$  (for any  $A$ ) is just a residue disk of the  $j$ -line. When  $A$  is a supersingular curve, it is well-known that  $W_A(p)$  is isomorphic, over  $\mathbb{Q}_{p^2} := W(\mathbb{F}_{p^2}) \otimes \mathbb{Q}_p$ , to an open annulus of width  $i(A) := |\text{Aut}(A)|/2$ . This means that one can choose a parameter  $x_A$  on  $W_A(p)$  over  $\mathbb{Q}_{p^2}$  which identifies it with the (open) annulus,  $A_{\mathbb{Q}_{p^2}}(p^{-i(A)}, 1)$ . In fact, we can and will always do this in such a way that  $v(x_A(E, C)) = i(A)h(E)$  when  $C = K_1(E)$  and

<sup>16</sup>Thinking of the kernel of reduction of  $E$  as a disk, the set of points of order  $p^n$  are not always equidistant from the identity. When  $E \in W_n$ ,  $K_n(E)$  is the union over  $i \leq n$  of those points of order  $p^i$  which are closest to the identity.

<sup>17</sup>Equivalently,  $j(E)$  is not congruent to a supersingular  $j$ -invariant modulo  $m_{\mathbb{R}_p}$ .

$i(A)(1-h(E/C))$  otherwise (this is justified using arguments as in [Bu, Theorem 3.3, §4]).

Now, inside the annulus,  $W_A(p)$ , there are three concentric circles which will be essential for our analysis of  $X_0(p^2)$  and  $X_0(p^3)$ . First there is the “too-supersingular circle,” which we denote by  $\mathbf{TS}_A$ , whose points correspond to pairs  $(E, C)$  where the canonical subgroup of  $E$  is trivial. Equivalently these are the points with  $h(E) \geq p/(p+1)$ . Next there is the “self-dual circle,” denoted by  $\mathbf{SD}_A$ , whose points correspond to pairs  $(E, C)$  where the subscheme  $C$  of order  $p$  is potentially self-dual, i.e., isomorphic to its Cartier dual after finite base extension. Equivalently,  $\mathbf{SD}_A$  consists of those points which satisfy  $h(E) = 1/2$  and  $C = K_1(E)$ . When  $A/\mathbb{F}_p$ ,  $\mathbf{SD}_A$  can also be described as the unique circle in  $W_A(p)$  which is fixed by the involution  $w_1$ , and hence we call it the “Atkin-Lehner circle.” Finally, we must also consider what might be called the “anti-Atkin-Lehner circle.” It is the subspace,  $\mathbf{C}_A \subseteq W_A(p)$ , whose points correspond to pairs  $(E, C')$  for which there exists a  $C$  such that  $(E, C) \in \mathbf{SD}_A$  but  $C' \neq C$ . We let  $\tau_f : \mathbf{C}_A \rightarrow \mathbf{SD}_A$  be the map which corresponds to replacing the cyclic subgroup  $C'$  with  $K_1(E)$ . Then  $\tau_f$  is rigid analytic since it is the restriction of  $s_1 \circ \pi_f$ .

**Remark 3.4.** *The fact that the above regions are circles follows from Buzzard’s discussion of rigid subspaces of  $X_1(p)$  in [Bu, §4]. Using a parameter  $x_A$  chosen as above, the circles  $\mathbf{TS}_A$ ,  $\mathbf{SD}_A$  and  $\mathbf{C}_A$  are those where  $v(x_A)/i(A)$  equals  $p/(p+1)$ ,  $1/2$ , and  $1 - 1/(2p)$ , respectively.*

From above, whenever  $A/\mathbb{F}_p$  is supersingular,  $W_A(p)$  is an annulus which is preserved by the Atkin-Lehner involution  $w_1$ , and which is mapped onto the residue disk  $W_A(1)$  via  $\pi_f$ . In our analysis of the stable models of  $X_0(p^2)$  and  $X_0(p^3)$ , we will need to work with fairly explicit approximations for the restrictions of  $\pi_f$  and  $w_1$  to these subspaces:

**Theorem 3.5.** *Let  $\mathbb{Z}_{p^2} := W(\mathbb{F}_{p^2})$  and  $A/\mathbb{F}_p$  be a supersingular curve with  $j(A) \neq 0, 1728$ . There are parameters  $s$  and  $t$  over  $\mathbb{Z}_{p^2}$  which identify  $W_A(1)$  with the disk  $B_{\mathbb{Q}_{p^2}}(1)$  and  $W_A(p)$  with the annulus  $A_{\mathbb{Q}_{p^2}}(p^{-1}, 1)$ , and series  $F(T), G(T) \in T\mathbb{Z}_{p^2}[[T]]$ , such that*

- (i)  $w_1^*(t) = \kappa/t$  for some  $\kappa \in \mathbb{Z}_{p^2}$  with  $v(\kappa) = 1$ .
- (ii)  $\pi_f^*s = F(t) + G(\kappa/t)$ , where
  - (a)  $F'(0) \equiv 1 \pmod{p}$ , and
  - (b)  $G(T) \equiv (F(T))^p \pmod{p}$ .

*Proof.* One only has to translate results in [dSh, §3]. Our  $t$  and  $\kappa$  are de Shalit’s  $y$  and  $\pi$ . Then our  $\pi_f^*s$  is  $\psi(y) - \beta_0$  in de Shalit’s language. The theorem follows from (4) of §2.2, as well as Lemma 1 and Corollaries 2-4 of §3 in *ibid.*  $\square$

Note that the parameter  $t$  from Theorem 3.5 is a suitable choice for  $x_A$ . This follows from condition (ii), which guarantees that  $\pi_f$  has degree  $p+1$  on the circle where  $v(t) = p/(p+1)$  and degree 1 or  $p$  on all other concentric circles.

### 3.2 Neighborhoods of the Ordinary Locus

The finitely many subspaces,  $W_A(p^n)$  (defined above) where  $A$  runs over supersingular curves over  $\mathbb{F}_{p^2}$ , cover the supersingular locus of  $X_0(p^n)$  over  $\mathbb{Q}_{p^2}$ , i.e., the subspace whose points over  $\mathbb{C}_p$  correspond to pairs  $(E, C)$  where  $E$  has supersingular reduction. Furthermore, these subspaces become connected wide opens over  $\mathbb{C}_p$ , by Theorem 2.29. We will now describe a finite collection of subspaces,  $W_{ab}^\pm \subseteq X_0(p^n)$ , which cover the ordinary locus. These will, in fact, be shown to be basic wide opens when  $n \leq 3$ , and we do expect this to hold more generally. Essentially, we extend the irreducible affinoids,  $\mathbf{X}_{ab}^\pm$  (introduced in [C3]<sup>18</sup>), to wide open neighborhoods, by considering points  $(E, C)$  which are *nearly* ordinary in the sense that either  $K(E)$  or  $K(E/C)$  is large.

More precisely, for  $a \geq b \geq 0$  with  $a + b = n$ , we start by letting

$$W_{ab} = \{ (E, C) : |K(E)| \geq p^n, |K(E) \cap C| = p^a \}.$$

For  $b > a \geq 0$  with  $a + b = n$ , we then define  $W_{ab} = w_n(W_{ba})$ . Now we show that the pairing on  $K_a(E)$ , which was defined in [C3] for points  $(E, C) \in W_{ab}$  where  $E$  has ordinary reduction, carries over to all points in  $W_{ab}$ . Let  $(E, C)$  be a point of  $W_{ab}$  with  $a \geq b$ , and let  $A, B \in K_a(E)$ . Then by the definition of  $W_{ab}$  we can choose  $P \in C$  and  $Q \in K_n(E)$  such that  $p^b P = A$  and  $p^b Q = B$ . Now set  $\mathcal{P}_{E,C}(A, B) = e_n(P, Q)$ , where  $e_n(\cdot, \cdot)$  denotes the Weil pairing on  $E[p^n]$ . This gives a well-defined pairing of  $K_a(E)$  with itself onto  $\mu_{p^b}$ . Furthermore, if  $p > 2$ , there are exactly two isomorphism classes of pairings on  $\mathbb{Z}/p^a\mathbb{Z}$  onto  $\mu_{p^b}$  whenever  $b > 0$ . Let  $e^+(\cdot, \cdot)$  and  $e^-(\cdot, \cdot)$  be representatives for these classes. Then, essentially repeating the argument for the ordinary affinoids,  $X_{ab}^\pm$ , from [C3], there is a rigid subspace  $W_{ab}^\pm$  of  $X_0(p^n)$  defined over  $\mathbb{Q}_p(\sqrt{(-1)^{(p-1)/2}})$  whose  $\mathbb{C}_p$  valued points are

$$\{ (E, C) \in W_{ab} \mid (K_a(E), \mathcal{P}_{E,C}) \cong (\mathbb{Z}/p^a\mathbb{Z}, e^\pm) \}.$$

Set  $W_{n0}^+ = W_{n0}^- = W_{n0}$ , and for  $b > a \geq 0$  we set

$$W_{ab}^\beta = w_n(W_{ba}^{\left(\frac{-1}{p}\right)^\beta}).$$

Thus,  $\mathbf{X}_{ab}^\pm$  is just the affinoid whose points are those  $(E, C) \in W_{ab}^\pm$  for which  $E$  has ordinary or multiplicative reduction. It is *not* immediate that  $W_{ab}^\pm$  is a basic wide open with  $\mathbf{X}_{ab}^\pm$  as a minimal underlying affinoid. We will show that this is the case, however, when  $n \leq 3$ , and we do expect it to hold for arbitrary  $n$  as well. The affinoid,  $\mathbf{X}_{ab}^\pm$ , is well understood from results of [C3]. In particular, we have the following result, which was proven but not made explicit in [C3].

**Proposition 3.6.** *The affinoid  $\mathbf{X}_{ab}^\pm$ , with  $a \geq b > 0$ , is defined and has good reduction over  $\mathbb{Q}_p(\mu_{p^b})$ .*

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<sup>18</sup>When  $a < b$ , the  $X_{ab}^\beta$  here is the same as  $X_{ab}^{\left(\frac{-1}{p}\right)^\beta}$  from [C3].

*Proof.* It was proven in [C3, §0] that  $\mathbf{X}_{ab}^\pm$  is an affinoid defined over the quadratic subfield of  $\mathbb{Q}_p(\mu_p)$ . For  $\zeta \in \mu_{p^b}$  we can define an embedding  $a_\zeta$  of  $\mathbf{X}_{ab}^\pm$  onto an affinoid in  $X_1(p^b)^{bal}$  by taking  $a_\zeta(E, C)$  to be the point which is represented by the balanced  $\Gamma_1(p^b)$ -structure (see [KM, (3.3)]):

$$P, E \quad \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\hat{\alpha}} \end{array} \quad E/C, P'.$$

Here we have  $P \in K_b(E)$ ,  $\mathcal{P}_{E,C}(P, P) = \zeta$ , and  $P' = \alpha(Q)$  for some  $Q \in E[p^b]$  such that  $(P, Q) = \zeta$ . This image affinoid reduces to  $Ig(p^b)$  by (the extension to level 1 of) [KM, pg. 450].<sup>19</sup>  $\square$

**Corollary 3.7.** *The affinoid  $\mathbf{X}_{ab}^\pm$ , with  $a + b = n$ , is defined and has good reduction over  $\mathbb{Q}_p(\mu_{p^{\lfloor n/2 \rfloor}})$ .*

*Proof.* When  $a \geq b$  this follows immediately from the proposition. Otherwise, apply  $w_n$  first.  $\square$

## 4 Formal Groups

In the previous section we defined a finite collection of connected wide opens,  $W_A(p^n)$ , which cover the supersingular locus of  $X_0(p^n)$ . Unfortunately,  $W_A(p^n)$  is only basic when  $n \leq 2$ . Therefore, in order to arrive at a stable covering of  $X_0(p^n)$ , it is necessary to use smaller subspaces of  $W_A(p^n)$ . One approach is to use canonical subgroup considerations as in Section 3.1. Another is to use the interpretation from [WH] of an elliptic curve, over a complete local ring  $R$  with residue characteristic  $p$ , as a lifting of some formal group in characteristic  $p$ . In particular this will enable us to use explicit formulas of Gross-Hopkins which we recall in Section 4.2.

**Theorem 4.1** (Woods-Hole Theory). *Suppose  $R$  is the ring of integers in a complete subfield of  $\mathbb{C}_p$ , with residue field  $\mathbb{F}$ . The category of elliptic curves over  $R$  is equivalent to the category of triples  $(F, A, \alpha)$ , where  $F$  is a formal group over  $R$ ,  $A$  is an elliptic curve over  $\mathbb{F}$  and  $\alpha : \bar{F} \rightarrow \hat{A}$  is an isomorphism. A morphism between two triples,  $(F, A, \alpha)$  and  $(F', A', \beta)$ , is a pair  $(\sigma, \tau)$ , where  $\sigma : F \rightarrow F'$  and  $\tau : A \rightarrow A'$  are homomorphisms such that the following diagram commutes.*

$$\begin{array}{ccc} \bar{F} & \xrightarrow{\bar{\sigma}} & \bar{F}' \\ \alpha \downarrow & & \downarrow \beta \\ \hat{A} & \xrightarrow{\hat{\tau}} & \hat{A}' \end{array}$$

*Proof.* If  $E$  is an elliptic curve over  $R$  let  $\mathcal{F}_R(E) = (\hat{E}, \bar{E}, \iota)$ , where  $\iota : \bar{E} \rightarrow \hat{E}$  is the natural isomorphism. This is a functor, compatible with changing  $R$ , from

<sup>19</sup> $Ig(p^b)$  is the Igusa curve in characteristic  $p$  which classifies pairs,  $(E, \psi)$ , where  $E$  is an elliptic curve and  $\psi : \mu_{p^b} \hookrightarrow E$  (studied in [I]).

the first category to the second. We claim this is an equivalence of categories. The analogous statement is proven when  $R$  is a local Artinian ring with residue field of characteristic  $p$  in §6 of [WH]. Then on page 7, line -6 of [WH], it is explained that by “passing to the limit. . . one sees that it continues to hold over a complete local Noetherian ring.” Thus the theorem is true when  $R$  is the ring of integers in a complete discretely valued subfield of  $\mathbb{C}_p$ .

To obtain it more generally, we apply Theorem 1 of [WH]. This theorem implies that given an elliptic curve  $A$  over an algebraic extension of  $\mathbb{F}_p$ , the collection of liftings of  $\hat{A}$  to  $\mathbb{R}_p$  is naturally the set of points in a wide open disk  $\mathcal{D}$ . On the other hand, the set of liftings of  $A$  to  $\mathbb{R}_p$  is the set of points in a residue disk  $\mathcal{R}$  of  $X(1)$  and  $\mathcal{F}$  yields a degree one rigid analytic map from  $\mathcal{R}$  to  $\mathcal{D}$  with dense image. Hence it is an isomorphism.  $\square$

In light of this theorem, we may think of points  $(E, C) \in W_A(p^n)$  as triples,  $(F, C, \alpha)$ , where  $F$  is a formal group,  $C \subseteq F$  is a cyclic subgroup of order  $p^n$ , and  $\alpha : \bar{F} \rightarrow \hat{A}$  is an isomorphism. We then refer to such a triple as a Woods Hole representation of  $(E, C)$ . There are two specific ways in which we apply this theory. First of all, from the fact that all supersingular elliptic curves are  $p$ -prime isogenous, we are able to show that all supersingular regions,  $W_A(p^n)$  (for a fixed  $p$  and  $n$ ), are *nearly* isomorphic. Along with the result in Appendix B, this enables us to do all of our calculations under the simplifying assumption that  $A/\mathbb{F}_p$  and  $j(A) \neq 0, 1728$ . Secondly, we make extensive use of the natural action of the  $p$ -adic group  $\text{Aut}(\hat{A})$  on  $W_A(p)$ , which was studied in great detail in [GH].

## 4.1 All Supersingular Regions are (nearly) Isomorphic

**Proposition 4.2.** *Let  $A, A'/\mathbb{F}_{p^2}$  be two supersingular elliptic curves, with  $j(A) \neq 0$  or 1728. Let  $\mathbb{F}/\mathbb{F}_{p^2}$  be a finite extension over which  $A$  and  $A'$  are  $p$ -prime isogenous (which always exists). Then the wide open  $W_{A'}(p^n)$  is isomorphic over  $W(\mathbb{F}) \otimes \mathbb{Q}_p$  to the quotient of  $W_A(p^n)$  by a faithful action of  $\text{Aut}(A')/\{\pm 1\}$ .*

*Proof.* Let  $\iota : A \rightarrow A'$  be an isogeny of degree prime to  $p$  over  $\mathbb{F}$ . Since  $(\deg \iota, p) = 1$ , the induced map  $\hat{\iota} : \hat{A} \rightarrow \hat{A}'$  is an isomorphism of formal groups. So in Woods Hole terms we may define a map  $\psi_\iota : W_A(p^n) \rightarrow W_{A'}(p^n)$  by taking

$$\psi_\iota(F, C, \alpha) = (F, C, \hat{\iota} \circ \alpha).$$

To show that the map is, in fact, well-defined, suppose that the triples  $(F_1, C_1, \alpha_1)$  and  $(F_2, C_2, \alpha_2)$  represent the same point of  $W_A(p^n)$ . This means that there are isomorphisms  $\gamma : F_1 \rightarrow F_2$  (mapping  $C_1$  to  $C_2$ ) and  $\tau : A \rightarrow A$  such that  $\alpha_2 \circ \bar{\gamma} = \hat{\tau} \circ \alpha_1$ . Because  $j(A) \neq 0, 1728$ , we know that  $\tau = \pm 1$ . Therefore  $\hat{\tau}$  commutes with all isogenies. In particular, composing with  $\hat{\iota}$  on both sides we have

$$\hat{\iota} \circ \alpha_2 \circ \bar{\gamma} = \hat{\tau} \circ \hat{\iota} \circ \alpha_1.$$

Therefore  $(F_1, C_1, \hat{\iota} \circ \alpha_1)$  and  $(F_2, C_2, \hat{\iota} \circ \alpha_2)$  are Woods Hole representations of the same point in  $W_{A'}(p^n)$ , and  $\psi_\iota$  is well-defined.

To show that  $\psi_\iota$  is onto, choose any point of  $W_{A'}(p^n)$  and let  $(F, C, \beta)$  be one of its Woods Hole representations (so  $\beta : \bar{F} \rightarrow \hat{A}'$  is an isomorphism). As  $\hat{\iota}$  is an isomorphism, we can choose a point of  $W_A(p^n)$  by taking  $(F, C, \hat{\iota}^{-1} \circ \beta)$ , and this point maps onto our chosen point of  $W_{A'}(p^n)$  by definition. (Note, however, that this does *not* define a map from  $W_{A'}(p^n)$  to  $W_A(p^n)$ , as our original choice of triple was non-canonical.)

Finally, suppose that two points of  $W_A(p^n)$ , represented by the triples,  $(F_1, C_1, \alpha_1)$  and  $(F_2, C_2, \alpha_2)$ , have the same image in  $W_{A'}(p^n)$ . Then there must be isomorphisms  $\gamma : F_1 \rightarrow F_2$  (taking  $C_1$  to  $C_2$ ) and  $\tau : A' \rightarrow A'$  such that the following hold:

$$\begin{aligned}\hat{\iota} \circ \alpha_2 \circ \bar{\gamma} &= \hat{\tau} \circ \hat{\iota} \circ \alpha_1 \\ \alpha_2 \circ \bar{\gamma} &= \hat{\text{id}} \circ (\hat{\iota}^{-1} \circ \hat{\tau} \circ \hat{\iota}) \circ \alpha_1\end{aligned}$$

In particular,  $\tau \mapsto ((F, C, \alpha) \mapsto (F, C, \hat{\iota}^{-1} \tau \hat{\iota} \circ \alpha))$  gives a faithful action of  $\text{Aut}(A')/\{\pm 1\}$  on the fibers of  $\psi$ .  $\square$

**Remark 4.3.** *Suppose now that  $\mathbb{F} \supseteq \mathbb{F}_{p^2}$  is a field over which all supersingular curves are  $p$ -prime isogenous. It follows, then, that all of the regions,  $W_A(p^n)$ , are nearly isomorphic over  $W(\mathbb{F}) \otimes \mathbb{Q}_p$ . We show in [CMc, Thm 5.5] that this  $\mathbb{F}$  can always be taken to be  $\mathbb{F}_{p^{24}}$ .*

## 4.2 Woods Hole Action and Gross-Hopkins Theory

The other way in which we use Woods Hole Theory is to define a continuous action of a  $p$ -adic group on  $W_A(p^n)$ . In particular, when  $A$  is a supersingular elliptic curve it is well-known (see Main Theorem in [T1], for example) that

$$B := \text{End}(\hat{A}) \cong \mathbb{Z}_p[i, j, k],$$

where  $i^2$  is a quadratic non-residue,  $j^2 = -p$ , and  $ij = -ji = k$ . Furthermore, we may take  $j$  to be the Frobenius endomorphism whenever  $A$  is defined over  $\mathbb{F}_p$ . Then  $B^* = \text{Aut}(\hat{A})$  acts on  $W_A(p^n)$  by

$$\rho(F, C, \alpha) = (F, C, \rho \circ \alpha) \quad \rho \in B^*.$$

**Remark 4.4.** *The subgroup,  $\mathbb{Z}_p^* \subseteq B^*$ , acts trivially on  $W_A(p^n)$ . Indeed, for  $\rho \in \mathbb{Z}_p^*$ , just take  $\sigma = \rho^{-1}$  and  $\tau = \text{id}$  in Theorem 4.1. Not only does this define an isomorphism between  $(F, \alpha)$  and  $(F, \rho \circ \alpha)$ , but in fact the isomorphism leaves invariant the subgroups of  $F$  of order  $p^n$ .*

Gross and Hopkins studied the analogous action for deformation spaces of finite height formal groups, and explicitly computed the action in the height two case in [GH, §25]. In order to better understand their results (and translate them into our setting), we now offer a brief review of their theory under suitable simplifying assumptions. First, let  $K$  be a finite unramified extension of  $\mathbb{Q}_p$  with residue field  $\mathbb{F} \supseteq \mathbb{F}_{p^2}$ , and let  $F_0/\mathbb{F}$  be a fixed height two formal group.

They show that there is a rigid space over  $K$ , denoted by  $X_K$ , whose  $L$ -valued points for any finite extension  $L$  of  $K$  correspond to liftings of  $F_0$  to a formal group over  $\mathcal{O}_L$ . Here two liftings are equivalent (say,  $(G_1, \gamma_1)$  and  $(G_2, \gamma_2)$  with  $\gamma_i : \overline{G}_i \xrightarrow{\sim} F_0$ ) if there is an isomorphism between them which induces the identity on  $F_0$ . Then  $\text{Aut}(F_0)$  acts (rigid-analytically) on  $X_K$  in the same manner as above, and Gross-Hopkins make this action completely explicit via their crystalline period mapping,

$$\Phi : X_K \rightarrow \mathbf{P}_K^1,$$

which can be understood as follows. Again, it is well-known that  $B := \text{End}(F_0)$  is isomorphic to the maximal order of some quaternion algebra over  $\mathbb{Q}_p$ , and hence  $B \otimes K$  is (non-canonically) isomorphic to  $M_{2 \times 2}(K)$ . Since the image of  $B^*$  in  $M_{2 \times 2}(K)$  must take lines to lines, we thus obtain an action of  $B^*$  on  $\mathbf{P}_K^1$ . Gross-Hopkins define the (rigid-analytic) map  $\Phi$  and decompose  $B \otimes K$  in such a way that  $\Phi(\rho x) = \Phi(x)^\rho$  for all  $\rho \in B^*$ , i.e.  $\Phi$  is  $B^*$ -equivariant. So the beauty of Gross-Hopkins theory is that the action of  $B^*$  on  $X_K$  can be concretely expressed in terms of linear algebra.

Indeed, suppose now that  $A/\mathbb{F}_p$  with  $j(A) \neq 0, 1728$ . Then  $X_K$  and  $W_A(1)$  are naturally isomorphic over the unramified quadratic extension  $K$  of  $\mathbb{Q}_p$ , and we may decompose  $B$  as  $R \oplus Rj$ , where  $R = \mathbb{Z}_p[i] \cong \mathcal{O}_K$  and  $j$  is the Frobenius endomorphism of  $A$  (as above). Then from [GH, §25],  $\rho = \alpha + j\beta \in B^*$  (with  $\alpha, \beta \in \mathbb{Z}_p[i]$ ) acts on  $\mathbf{P}^1(K) = \Phi(X_K)$  via multiplication on the **right** by the matrix,

$$\begin{bmatrix} \alpha & -p\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix}. \quad (1)$$

Of course, this formula only completely defines the action of  $B^*$  on  $B^*$ -stable subspaces of  $W_A(1)$  on which  $\Phi$  is an injection (for example, the canonical liftings of Gross [G, 2.1]). In [GH, 25.12], the authors specify an affinoid disk  $Y \subseteq W_A(1)$  for which this is the case, and which maps via  $\Phi$  onto the disk  $v(t) \geq 1/p$  where  $t$  is the parameter on  $\mathbf{P}^1$  corresponding to the row vector  $[1, t]$ . This parameter is distinct from the parameter on  $W_A(p)$  from Theorem 3.5. However, from the explicit action of  $B^*$  on  $\Phi(Y)$  and the fact that this  $t$  vanishes at some canonical lifting (which is necessarily too-supersingular), there is significant compatibility between the two. In particular, it is clear that the canonical section of  $\pi_f : W_A(p) \rightarrow W_A(1)$  exists over the annulus in  $Y \subseteq W_A(1)$  described by  $1/p < v(t) < p/(p+1)$  and preserves valuations with respect to the two parameters. As  $B^*$  acts equivariantly with respect to  $\pi_f$ , the upshot of all this is that  $B^*$  acts on the sub-annulus of  $W_A(p)$  which is identified via  $\Phi \circ \pi_f$  with the annulus,  $\frac{1}{p} < v(t) < \frac{p}{p+1}$ , according to

$$\rho(t) = \frac{-p\bar{\beta} + \bar{\alpha}t}{\alpha + \beta t}, \quad \rho = \alpha + j\beta. \quad (2)$$

In particular, we are most interested in the action of  $B^*$  on the Atkin-Lehner circle (equivalently, where  $v(t) = 1/2$ ). The following proposition and remark

summarize the specific results (still assuming that  $A/\mathbb{F}_p$  and  $j(A) \neq 0, 1728$ ) which we will need for our explicit analysis of  $X_0(p^3)$ .

**Definition 4.5.** For  $\rho = \alpha + j\beta$  (as above), let  $\rho' = \bar{\alpha} + j\bar{\beta}$ , and let  $B'$  be the set of all  $\rho \in B^*$  such that  $\rho\rho' \in \mathbb{Z}_p^*$ . Alternatively,  $B'$  is just the set of all  $\rho \in B^*$  with  $\rho = a + bi + dk$ .

**Proposition 4.6.** Suppose that  $j(A) \neq 0, 1728$ . For any  $\rho \in B^*$ , let  $w_\rho := \rho \circ w_1$ . Then  $w_\rho$  is an automorphism of  $\mathbf{SD}_A$  with two fixed points, and an involution exactly when  $\rho \in B'$ .

*Proof.* If  $(E_2, K(E_2)) = w_1(E_1, K(E_1))$  are two points of  $\mathbf{SD}_A$ , this means that there is a degree  $p$  isogeny  $f : E_1 \rightarrow E_2$  with kernel  $K(E_1)$ . Since  $A$  is supersingular with  $\text{Aut}(A) = \pm 1$ ,  $f$  can only induce  $\pm j$  in  $\text{End}(A)$  (and hence in  $B = \text{End}(\hat{A})$ ). Now,  $j \notin B^*$ , but the full group,  $(B \otimes K)^\times$ , acts equivariantly on  $\Phi(X_K)$  by [GH, 23.11]. So this means that on  $\mathbf{SD}_A$  we may identify  $w_1$  with  $\pm j$  (and the sign is irrelevant).

To determine when  $w_\rho$  is an involution, we first verify that  $\rho \circ w_1 = w_1 \circ \rho'$  (equivalently,  $\rho j = j\rho'$ ). This shows that  $w_\rho^2$  acts like  $\rho\rho'$ , and only  $\mathbb{Z}_p^* \subseteq B^*$  acts trivially from (2). So  $w_\rho$  is an involution exactly when  $\rho \in B'$ . In particular,  $w_\rho$  is given by

$$w_\rho(t) = \frac{-p\bar{\alpha} - p\bar{\beta}t}{-p\beta + \alpha t}, \quad (3)$$

and the explicit formula shows that in any case  $w_\rho$  has two fixed points.  $\square$

**Remark 4.7.** To better understand how  $\rho \in B^*$  and  $w_\rho$  act on  $\overline{\mathbf{SD}}_A$ , we could choose the parameter  $u = t/\sqrt{-p}$  which identifies  $\mathbf{SD}_A$  with  $C[1]$ . Then, by reducing equations (2) and (3) from above, on  $\overline{\mathbf{SD}}_A \cong \mathbf{G}_m$  we have

$$\rho\bar{u} = \bar{\alpha}\alpha^{-1}\bar{u} = \zeta\bar{u} \quad \text{and} \quad w_\rho\bar{u} = \frac{\bar{\alpha}\alpha^{-1}}{\bar{u}} = \frac{\zeta}{\bar{u}},$$

for some  $\zeta \in \mu_{p+1} \subseteq \mathbb{F}_{p^2}^*$ . So on  $\overline{\mathbf{SD}}_A$ , the  $w_\rho$ 's reduce to  $p+1$  distinct involutions with  $2(p+1)$  distinct fixed points (in a  $\mu_{2(p+1)}$  orbit). Furthermore, each of these involutions of  $\overline{\mathbf{SD}}_A$  lifts to an involution of  $\mathbf{SD}_A$ .

Another way to think of the fixed points of the automorphisms,  $\{w_\rho\}$ , is that they correspond to elliptic curves whose formal groups have complex multiplication by the ring of integers in a ramified quadratic extension of  $\mathbb{Q}_p$  (see Proposition 4.9 below). This point of view becomes crucial when we determine the field of definition of our stable model, because it ties our construction to the arithmetic theory of CM elliptic curves. To this end, we make the following definition.

**Definition 4.8.** For  $K$  a complete subfield of  $\mathbb{C}_p$ , an elliptic curve  $E/K$  has fake CM if  $\text{End}_K \hat{E} \neq \mathbb{Z}_p$  and potential fake CM if  $\text{End}_{\mathbb{C}_p} \hat{E} \neq \mathbb{Z}_p$ .

**Proposition 4.9.** *Let  $(E, C)$  be any point of  $\mathbf{SD}_A$ . Then the following statements are equivalent.*

- (i)  $(E, C)$  is fixed by  $w_\rho$  for some  $\rho \in B'$
- (ii)  $(E, C)$  is fixed by  $w_\rho$  for some  $\rho \in B^*$
- (iii)  $E$  has potential fake CM by  $\mathbb{Z}_p[\pi]$  where  $\pi \in \text{End}(\hat{E})$  and  $C = \ker \pi$ .

*Proof.* We will show that (ii) is equivalent to both (i) and (iii) with a Woods Hole argument. So before we begin we must reinterpret condition (ii), that  $(E, C)$  is fixed by  $w_\rho$ , in the language of Theorem 4.1. Let  $(F, \alpha, C)$  be a Woods Hole representation of  $(E, C)$ , and let  $\iota_C : F \rightarrow F/C$  be the natural map. Then  $(E, C)$  is a fixed point of  $w_\rho$  if and only if there is an isomorphism,  $\sigma : F/C \rightarrow F$ , which makes the following diagram commute.

$$\begin{array}{ccccccc}
 \bar{F} & \xrightarrow{\iota_C} & \overline{F/C} & \xrightarrow{id} & \overline{F/C} & \xrightarrow{\bar{\sigma}} & \bar{F} \\
 \alpha \downarrow & & \beta \downarrow & & \rho \circ \beta \downarrow & & \alpha \downarrow \\
 \hat{A} & \xrightarrow{j} & \hat{A} & \xrightarrow{\rho} & \hat{A} & \xrightarrow{id} & \hat{A}
 \end{array}$$

Note that the first commuting square represents the isogeny (of elliptic curves),  $E \rightarrow E/C$ . The pair,  $(F/C, \rho \circ \beta)$ , then corresponds to the elliptic curve,  $\rho(E/C)$ .

Now, to show (iii) implies (ii), suppose first that we are given  $\pi \in \text{End}(F)$  with  $\ker(\pi) = C$ . Then  $\pi$  must factor as  $\sigma \circ \iota_C$  for some isomorphism,  $\sigma : F/C \rightarrow F$ , and we may take  $\rho = \alpha \circ \bar{\sigma} \circ \beta^{-1} \in B^*$  in the above diagram. Conversely, if  $(E, C)$  is a fixed point of  $w_\rho$  for some  $\rho = a + bi + cj + dk \in B^*$ , and hence we have a commutative diagram as above,  $\text{End}(F)$  must contain both  $\pi := \sigma \circ \iota_C$  and  $\pi_0 := \pi + pc$ . Using the diagram to compute inside  $\text{End}(\hat{A})$ , we then have

$$\alpha \circ \bar{\pi}_0 \circ \alpha^{-1} = \rho j - cj^2 = (\rho - cj)j.$$

Note that  $\rho - cj \in B'$ . Therefore  $\pi_0^2 \in p\mathbb{Z}_p^*$ , which means that  $\mathbb{Z}_p[\pi_0] = \mathbb{Z}_p[\pi]$  is already the maximal order in a ramified quadratic extension of  $\mathbb{Q}_p$  (and hence all of  $\text{End}(F)$ ). Thus we have shown that (ii) implies (iii). We also get for free, however, that (ii) implies (i), since  $(E, C)$  is now also fixed by  $w_{\rho_0}$  where  $\rho_0 := \rho - cj \in B'$ .  $\square$

**Corollary 4.10.** *If  $(E, C) \in \mathbf{SD}_A$  is fixed by  $w_{\rho_0}$  for some  $\rho_0 \in B'$ , then  $w_\rho$  fixes  $(E, C)$  precisely when  $\rho = a\rho_0 + bj$  for  $a \in \mathbb{Z}_p^*$  and  $b \in \mathbb{Z}_p$ .*

**Remark 4.11.** *With notation as above, suppose that  $(E, C) \in \mathbf{SD}_A$  is fixed by  $w_\rho$  and  $H$  is one of the non-canonical subgroups of  $E$  of order  $p$  (so  $(E, H) \in \mathbf{C}_A$ ). Since  $\pi_0^2 \in p\mathbb{Z}_p^*$  and  $\ker(\pi_0) = C$ , we must have  $\pi_0(H) = C$ . This implies that  $a + b\pi_0 \in (\mathbb{Z}_p[\pi_0])^* \cong \text{Aut}(F)$  fixes the non-canonical subgroups of order  $p$  when  $p|b$ , and transitively permutes them otherwise.*

**Remark 4.12.** *We show in [CMc, §3, Rmk 3.11] that the points which satisfy the conditions of Proposition 4.9 are precisely the canonical liftings of  $\hat{A}$  in the sense of [G, 2.1], where  $K$  is one of the ramified quadratic extensions of  $\mathbb{Q}_p$  and  $\hat{A}$  is given the structure of a formal  $\mathcal{O}_K$ -module.*

## 5 Stable Reduction of $X_0(p^2)$

At this point we have put down enough groundwork to prove a rigid analytic reformulation of Edixhoven's result on the stable reduction of  $X_0(p^2)$  (see [E, Thm 2.1.2, §2.5.1]). Most of the work is in computing the reduction of  $\mathbf{Y}_A$ , the underlying affinoid of  $W_A(p^2)$ . This is done by first embedding  $\mathbf{Y}_A$  into the product of two circles (specifically  $\mathbf{TS}_A \times \mathbf{TS}_A$ ) and then applying the explicit formula of Theorem 3.5. After that, we use results from Section 2 to show that the wide opens in

$$\{W_{20}, W_{11}^+, W_{11}^-, W_{02}\} \cup \{W_A(p^2) : A \text{ supersingular}\}$$

intersect properly and comprise a stable covering of  $X_0(p^2)$ .

**Lemma 5.1.** *Let  $\mathbf{Y}_A = \pi_\nu^{-1}(\mathbf{TS}_A)$ . If  $A/\mathbb{F}_p$ ,  $\mathbf{Y}_A$  is naturally isomorphic to*

$$S := \{ (x, y) \in \mathbf{TS}_A \times \mathbf{TS}_A \mid x \neq y, \pi_f(x) = \pi_f(y) \}.$$

*Proof.* If  $(x, y) \in S$ ,  $x = (E, C_1)$  and  $y = (E, C_2)$  for  $E$  some too-supersingular curve, and  $C_1$  and  $C_2$  are two distinct subgroups of order  $p$ . So we can define a map,  $\psi : S \rightarrow \mathbf{Y}_A$  by taking  $(x, y)$  to  $(E/C_1, p^{-1}C_2/C_1)$ . It is immediate that this takes values in  $\mathbf{Y}_A$  since  $\pi_\nu \circ \psi(x, y)$  is then

$$(E/E[p], p^{-1}C_2/E[p]) \cong (E, C_2).$$

Furthermore, we can define a map going the other way, say  $\phi$ , by taking  $(E, C) \in \mathbf{Y}_A$  to the pair  $(x, y) \in S$  with  $x = (E/pC, E[p]/pC)$  and  $y = (E/pC, C/pC)$ . This takes values in  $S$  precisely because  $\pi_\nu(E, C) = (E/pC, C/pC) \in \mathbf{TS}_A$ , and it is straightforward to check that  $\psi \circ \phi$  and  $\phi \circ \psi$  are the respective identities.  $\square$

**Proposition 5.2.** *Let  $A$  be as in Theorem 3.5. Then if  $K$  is any extension of  $W(\mathbb{F}_{p^2}) \otimes \mathbb{Q}_p$  such that  $(p+1) \mid e(K)$ ,  $\overline{\mathbf{Y}}_A := \overline{(\mathbf{Y}_A)}_K$  is a smooth, affine curve of genus  $(p-1)/2$  with 4 points at infinity (equation given below).*

*Proof.* Let  $x$  and  $y$  be parameters on  $\mathbf{TS}_A$  which are specializations of the parameter  $t$  on  $W_A(p)$  from Theorem 3.5. Then by Lemma 5.1,  $\mathbf{Y}_A$  can be described by the equation,

$$F(x) + G(\kappa/x) = F(y) + G(\kappa/y),$$

where  $v(x) = v(y) = p/(p+1)$ . Now choose any  $\alpha \in K$  with  $v(\alpha) = 1/(p+1)$  and substitute  $u = \alpha^p/x$  and  $v = \alpha^p/y$  into the above equation for  $\mathbf{Y}_A$  (so that  $v(u) = v(v) = 0$ ). Dividing through by  $\alpha^p$  we obtain an equation for  $\mathbf{Y}_A$  in  $u$  and  $v$  which has integral coefficients and satisfies the following congruence.

$$u^{-1} - v^{-1} \equiv (v^p - u^p)(\kappa/\alpha^{p+1})^p \pmod{\alpha}$$

Now let  $b = (\kappa/\alpha^{p+1})^p$  (a unit), and we obtain

$$1 \equiv buv(v - u)^{p-1} \pmod{\alpha}$$

as an equation for  $\overline{\mathbf{Y}}_A$ .

Strictly speaking, the above curve has three infinite points, with projective coordinates  $(0 : 1 : 0)$ ,  $(1 : 0 : 0)$ , and  $(1 : 1 : 0)$ . However, while the first two are nonsingular, the third splits into two points in the normalization. The genus can easily be computed by applying Riemann-Hurwitz to the equation,

$$s^{p+1} = \frac{b}{4}(r^2 - 1),$$

where  $s = 1/(v - u)$  and  $r = (v + u)/(v - u)$ .  $\square$

**Theorem 5.3.** *Let  $p \geq 13$  be a prime, and  $K$  an extension of  $W(\mathbb{F}_{p^{24}}) \otimes \mathbb{Q}_p(\mu_p)$  with  $(p + 1) \mid e(K)$ . The following is a semi-stable covering of  $X_0(p^2)$  over  $K$ .*

$$\mathcal{C}_0(p^2) := \{W_{20}, W_{11}^+, W_{11}^-, W_{02}\} \cup \{W_A(p^2) : A \text{ supersingular}\}$$

The affinoids  $\mathbf{X}_{ab}^\pm$  and  $\mathbf{Y}_A$  are minimal underlying affinoids in  $W_{ab}^\pm$  and  $W_A(p^2)$ .

*Proof.* The wide opens,  $W_{ab}^\pm$ , which cover the ordinary locus, are disjoint from each other, and we have

$$\mathbf{Y}_A = W_A(p^2) \setminus \bigcup W_{ab}^\pm.$$

Also, by Proposition 3.6, all four ordinary affinoids have good reduction over  $K$ . Therefore, it suffices to show that each  $\mathbf{Y}_A$  has good reduction, and that  $W_A(p^2) \cap W_{ab}^\pm$  is always an annulus.

First we demonstrate that the wide open intersections are annuli over  $K$  (where we still assume  $(p + 1) \mid e$ ). In the case of  $W_{20}$  this is immediate, as  $W_{20} \cap W_A(p^2)$  maps isomorphically onto the annulus over  $K$ ,

$$x_A^{-1}(A(p^{-i(A)/(p+1)}, 1)) \subseteq W_A(p),$$

via  $\pi_f$ . Similarly,  $W_{11}^\pm \cap W_A(p^2)$  maps onto the same annulus via  $\pi_f$ , but with degree  $(p - 1)/2$ . Then Theorem 2.6 implies that this too is an annulus over  $K$ . Finally,  $W_{02} \cap W_A(p^2)$  must be an annulus since it is isomorphic to the region,  $W_{20} \cap W_{A^\sigma}(p^2)$ , by the Atkin-Lehner involution  $w_2$ .

Next we consider the reductions of the affinoids  $\mathbf{Y}_A$ . Since  $p \geq 13$ , Theorem B.1 guarantees us a supersingular elliptic curve  $A_0$  for which Proposition 5.2 directly applies. Then for any other supersingular curve  $A$  we use Proposition 4.2. In particular, we choose a surjection,  $\psi_\iota$ , which maps  $W_{A_0}(p^2)$  onto  $W_A(p^2)$  with degree  $i(A)$ . If  $i(A) = 1$  the two regions are isomorphic and we're done. In any case,  $\psi_\iota$  necessarily takes  $\mathbf{Y}_{A_0}$  to  $\mathbf{Y}_A$  and is étale. Therefore  $\overline{\mathbf{Y}}_A$  is isomorphic to the quotient of  $\overline{\mathbf{Y}}_{A_0}$  by an automorphism of degree  $i(A)$  (which fixes the four infinite points). Hence  $\mathbf{Y}_A$  has good reduction and we're done.  $\square$

**Corollary 5.4.** *For any supersingular curve  $A$ , the reduction of  $\mathbf{Y}_A$  must have (with the correct choice of parameters) the equation,*

$$y^{\frac{(p+1)}{i(A)}} = x^2 - 1,$$

and genus  $(p + 1)/(2i(A)) - 1$ .

*Proof.* After a change of coordinates, the reduction of  $\mathbf{Y}_{A_0}$  has the equation,  $y^{p+1} = x^2 - 1$ , with two of the four infinite points moved to  $(\pm 1, 0)$  and two still at infinity. Now, any automorphism of order  $i(A)$  which acts on this curve and fixes these four points must fix  $x$  and take  $y$  to  $\zeta y$  where  $\zeta^{i(A)} = 1$ .  $\square$

**Remark 5.5.** Let  $K$  be as in Theorem 5.3 with  $e_p(K) = (p^2 - 1)/2$ . By computing the widths of the annuli in the stable covering (see Section 9.1 for more details), one finds intersection multiplicities of  $i(A)$  where  $\mathbf{X}_{11}^\pm$  meets  $\mathbf{Y}_A$  and  $i(A) \cdot (p - 1)/2$  where  $\mathbf{X}_{20}$  and  $\mathbf{X}_{02}$  meet  $\mathbf{Y}_A$ .

The following implies Theorem 3.1 of [C3].

**Corollary 5.6.** The point  $(E, C)$  is not in  $S := W_{20} \cup W_{11}^+ \cup W_{11}^- \cup W_{02}$  if and only if  $pC = K_2(E)$  and  $E[p]/pC = K_2(E/C)$ , or equivalently  $K_1(E/pC) = 0$ .

*Proof.*  $(E, C)$  is not in  $S$  if and only if it is in some  $\mathbf{Y}_A$ , which by definition means that  $E/pC$  has trivial canonical subgroup. This is equivalent to  $pC = K_2(E)$  and  $E[p]/pC = K_2(E/C)$  by [Bu, Theorem 3.3 (vi)].  $\square$

**Corollary 5.7.** The Hecke correspondence  $T_\ell$  takes a divisor supported on  $S$  to a divisor supported on  $S$  if and only if  $\ell \neq p$ .

*Proof.* This follows from the fact that

$$T_\ell(E, C) = \sum_{\substack{\deg \alpha = \ell \\ |\alpha C| = |C|}} (\alpha E, \alpha C).$$

$\square$

**Remark 5.8.** Using the fact that  $X(p) \cong X_0(p^2) \times_{X_0(p)} X_1(p)$ , Jared Weinstein and the first author have used the results of this section to determine a stable model of  $X(p)$ .

## 6 Outline of $X_0(p^3)$ Analysis

At this point we would like to construct a stable covering for  $X_0(p^3)$  in much the same way as was just done for  $X_0(p^2)$ . By analogy, the natural starting point would be the covering consisting of:

$$\{W_{30}, W_{21}^+, W_{21}^-, W_{12}^+, W_{12}^-, W_{03}\} \cup \{W_A(p^3) : A \text{ supersingular}\}.$$

This is not stable, however, because  $W_A(p^3)$  is not a basic wide open. This can actually be seen immediately from the fact that each  $W_A(p^3)$  at least contains the affinoids,  $\mathbf{E}_{1A} := \pi_f^{-1}(\mathbf{Y}_A)$  and  $\mathbf{E}_{2A} := \pi_\nu^{-1}(\mathbf{Y}_{A^\sigma})$  (which are nontrivial from Section 5). So our covering for  $X_0(p^3)$  must at least be refined to take these regions into account. In fact, things are much more complicated.

For simplicity, suppose that  $A/\mathbb{F}_p$  with  $j(A) \neq 0, 1728$  (other  $W_A(p^3)$ 's can be handled by Proposition 4.2). Since  $\pi_{11}$  maps  $W_A(p^3)$  onto the width 1 annulus,  $W_A(p)$ , this gives us a convenient way to keep track of where various subspaces are in relation to each other. For example, it follows from Section 5 that the above affinoids,  $\mathbf{E}_{1,A}$  and  $\mathbf{E}_{2,A}$ , lie over the circles described by  $v(x_A) = \frac{p}{p+1}$  and  $v(x_A) = \frac{1}{p+1}$  respectively (with parameter  $x_A$  as in Section 3). The former is the too-supersingular circle, and the latter is what was called the nearly too-supersingular circle in [C3, §3]. Lying in between these two circles is the Atkin-Lehner circle,  $\mathbf{SD}_A$ , where  $v(x_A) = 1/2$ . So lying “between”  $\mathbf{E}_{1,A}$  and  $\mathbf{E}_{2,A}$  in some sense is the affinoid,  $\mathbf{Z}_A := \pi_{11}^{-1}(\mathbf{SD}_A)$ . It turns out that this affinoid is where all of the new complication arises at the  $p^3$  level. We now give a brief summary of the analysis of  $\mathbf{Z}_A$  which will follow in Sections 7 and 8.

Much of our analysis of  $\mathbf{Z}_A$  is explicit (Section 8), and is based on an embedding into the product of two circles as in Lemma 5.1. More specifically, let  $\tau_f : \mathbf{C}_A \rightarrow \mathbf{SD}_A$  be as in Section 3. Then  $\mathbf{Z}_A$  can be identified with

$$S := \{ (x, y) \in \mathbf{C}_A \times \mathbf{C}_A \mid \tau_f(x) = w_1 \circ \tau_f(y) \}.$$

Now, since  $\pi_f \circ \tau_f = \pi_f$ , this identification along with de Shalit’s result (Theorem 3.5) gives us a way to explicitly compute the reduction of  $\mathbf{Z}_A$  as

$$X^{p+1} + X^{-(p+1)} = Z^p.$$

So  $\overline{\mathbf{Z}}_A$  has  $2(p+1)$  cuspidal singular points, and its normalization is a copy of the affine line whose completion is what we will eventually call a “bridging component.” Basically, we want to show that the  $2(p+1)$  singular residue classes of  $\mathbf{Z}_A$  are basic wide open subspaces, with underlying affinoids that reduce to  $y^2 = x^p - x$ .

To motivate and explain this, consider the identity,  $\pi_{11} \circ w_3 = w_1 \circ \pi_{11}$ , relating the Atkin-Lehner involutions on  $X_0(p^3)$  and  $X_0(p)$ . It follows immediately that  $w_3$  preserves  $\mathbf{Z}_A$ , as well as  $\tilde{\mathcal{D}} := \pi_{11}^{-1}(\mathcal{D})$  where  $\mathcal{D}$  is either of the residue disks of  $\mathbf{SD}_A$  preserved by  $w_1$ . Furthermore, a moduli-theoretic argument shows that  $w_3$  has  $2p$  fixed points which lie  $p : 1$  over the  $w_1$  fixed points in  $\mathbf{SD}_A$ . So  $\tilde{\mathcal{D}} \subseteq \mathbf{Z}_A$  is a wide open with one end upon which the involution  $w_3$  acts with  $p$  fixed points. We show that  $\tilde{\mathcal{D}}$  is in fact isomorphic to the complement of an affinoid disk near infinity in a hyper-elliptic curve which reduces to  $y^2 = x^p - x$  ( $w_3$  is the hyper-elliptic involution). Such an argument, however, would only account for two of the singular residue classes of  $\mathbf{Z}_A$ . In order to handle all of them, we use the action of  $B^* = \text{Aut}(\hat{A})$  to generalize the pair,  $(w_1, w_3)$ , to a pair,  $(w_\rho, \tilde{w}_\rho)$ , as was done in Proposition 4.6. Thus we are able to handle all  $2(p+1)$  residue classes because of Remark 4.7.

Once we have actually constructed all of the nontrivial components in the stable reduction of  $X_0(p^3)$ , the argument is reduced to showing that nothing else interesting can happen. We do this in Section 9, with a total genus calculation playing a key role. Again we first use the fact that all supersingular regions are (nearly) isomorphic along with the result of Appendix B, so that calculations only need to be done for a supersingular curve with  $A/\mathbb{F}_p$  and  $j(A) \neq 0, 1728$ .

The remaining cases of  $p \leq 11$  will be handled explicitly in [CMc, §6], which we hope will make our construction more understandable (as well as complete the main theorem).

## 7 The Bridging Component

Fix a supersingular elliptic curve  $A/\mathbb{F}_p$  with  $j(A) \neq 0, 1728$ . In this section we begin our analysis of the affinoid,  $\mathbf{Z}_A := \pi_{11}^{-1}(\mathbf{SD}_A) \subseteq W_A(p^3)$ . In particular, we show with a moduli-theoretic argument that  $\mathbf{Z}_A$  can be embedded into  $\mathbf{C}_A \times \mathbf{C}_A$ . Using the embedding, we then construct a family of involutions on  $\mathbf{Z}_A$ . These involutions are compatible (with respect to  $\pi_{11}$ ) with the involutions of  $\mathbf{SD}_A$  which were introduced in Proposition 4.6.

**Proposition 7.1.** *Let  $\mathbf{C}_A$  and  $\tau_f : \mathbf{C}_A \rightarrow \mathbf{SD}_A$  be as in Section 3. There is a natural isomorphism  $\psi$  from*

$$S := \{ (x, y) \in \mathbf{C}_A \times \mathbf{C}_A \mid \tau_f(x) = w_1 \circ \tau_f(y) \}$$

to  $\mathbf{Z}_A$ , such that  $w_3(\psi(x, y)) = \psi(y, x)$  and  $\pi_{11}(\psi(x, y)) = \tau_f(x)$ .

*Proof.* Suppose  $(x, y) \in S$ . Then there exists an  $(E, C) \in \mathbf{SD}_A$  such that  $x = (E, H)$  for some  $H \neq C$ . The  $p$  non-canonical subgroups of  $E/C$  are precisely the subgroups  $D/C$ , where  $D \subseteq E$  is a cyclic subgroup of order  $p^2$  with  $pD = C$  (see [Bu, 3.3]). Therefore, since  $\tau_f(x) = w_1(\tau_f(y)) = (E, C)$ , there is a unique  $D$  such that  $y = (E/C, D/C)$ . Hence we can define a map  $\psi : S \rightarrow W_A(p^3)$  by

$$\psi(x, y) = (E/H, (p^{-1}D)/H).$$

Note that  $(p^{-1}D)/H$ , and hence  $\psi$ , is well-defined since  $pD = C$  and  $H$  span  $E[p]$ . The key fact to check is that  $\psi(x, y)$  lies in  $\mathbf{Z}_A$ , i.e.  $\pi_{11}(\psi(x, y)) \in \mathbf{SD}_A$ .

$$\begin{aligned} \pi_{11}(E/H, (p^{-1}D)/H) &= (E/\langle H, pD \rangle, D/\langle H, pD \rangle) \\ &= (E/E[p], D/E[p]) \\ &\equiv (E, pD) = (E, C) \in \mathbf{SD}_A. \end{aligned}$$

This calculation shows that  $\psi(x, y) \in \mathbf{Z}_A$ , that  $\pi_{11}(\psi(x, y)) = \tau_f(x)$ , and more. Once a point  $(E, C) \in \mathbf{SD}_A$  is fixed, there are  $p$  independent choices for both  $H$  and  $D$ . Therefore we have produced  $p^2$  points of  $\mathbf{Z}_A$  which are in the image of  $\psi$  and in the  $\pi_{11}$ -fiber over that particular  $(E, C) \in \mathbf{SD}_A$ . Since the total degree of  $\pi_{11} : X_0(p^3) \rightarrow X_0(p)$  is only  $p^2$ , we can conclude that  $\psi$  maps *onto*  $\mathbf{Z}_A$ , and hence is an isomorphism. We now describe its inverse. For an arbitrary  $(E, K) \in \mathbf{Z}_A$ , let  $x(E, K) = (E/p^2K, E[p]/p^2K)$ ,  $y(E, K) = (E/pK, K/pK)$ , and  $\phi(E, K) = (x(E, K), y(E, K))$ . To show that  $\phi = \psi^{-1}$ , it suffices to check that  $\phi \circ \psi$  is the identity on  $S$  (with notation as

above).

$$\begin{aligned}
x(E/H, (p^{-1}D)/H) &= (E/\langle H, C \rangle, p^{-1}H/\langle H, C \rangle) \\
&= (E/E[p], p^{-1}H/E[p]) \equiv (E, H) \\
y(E/H, (p^{-1}D)/H) &= (E/\langle H, D \rangle, p^{-1}D/\langle H, D \rangle) \\
&= (E/\langle E[p], D \rangle, p^{-1}D/\langle E[p], D \rangle) \\
&\equiv (E/pD, D/pD) = (E/C, D/C).
\end{aligned}$$

Now that we have determined  $\psi^{-1}$ , we can verify the claim regarding  $w_3$  by applying  $\psi^{-1} \circ w_3 \circ \psi$  to the pair  $(x, y)$  where  $x = (E, H)$  and  $y = (E/C, D/C)$ .

$$\begin{aligned}
w_3 \circ \psi(x, y) &= w_3(E/H, (p^{-1}D)/H) \\
&= (E/\langle H, p^{-1}D \rangle, p^{-3}H/\langle H, p^{-1}D \rangle) \\
&= (E/\langle E[p], p^{-1}D \rangle, p^{-3}H/\langle E[p], p^{-1}D \rangle) \\
&\equiv (E/D, p^{-2}H/D) \\
x(E/D, p^{-2}H/D) &= (E/\langle D, H \rangle, p^{-1}D/\langle D, H \rangle) \\
&= (E/\langle E[p], D \rangle, p^{-1}D/\langle E[p], D \rangle) \\
&\equiv (E/pD, D/pD) = (E/C, D/C) = y \\
y(E/D, p^{-2}H/D) &= (E/\langle D, p^{-1}H \rangle, p^{-2}H/\langle D, p^{-1}H \rangle) \\
&= (E/E[p^2], p^{-2}H/E[p^2]) \equiv (E, H) = x
\end{aligned}$$

□

**Proposition 7.2.** *For each  $\rho \in B^*$ , we can define an automorphism  $\tilde{w}_\rho$  of  $\mathbf{Z}_A$  (identified with  $S$ ) by*

$$\tilde{w}_\rho(x, y) = (\rho y, \rho' x).$$

*Furthermore,  $\tilde{w}_\rho$  is compatible with  $w_\rho$ , in the sense that  $\pi_{11} \circ \tilde{w}_\rho = w_\rho \circ \pi_{11}$ , and is an involution of  $\mathbf{Z}_A$  whenever  $\rho \in B'$ .*

*Proof.* The action of  $B^*$  on  $W_A(p)$  preserves circles. So at least this defines a map from  $\mathbf{C}_A \times \mathbf{C}_A$  to itself. In order to verify that it preserves the subspace  $S$  we need to check that

$$\tau_f(x) = w_1 \circ \tau_f(y) \Rightarrow \tau_f(\rho y) = w_1 \circ \tau_f(\rho' x).$$

But  $\tau_f$  commutes with  $B^*$ . So this follows from the identity,  $\rho w_1 = w_1 \rho'$ , which was shown in the proof of Proposition 4.6.

By Remark 4.4, the inverse of  $\tilde{w}_\rho$  is given by  $\tilde{w}_\xi$  for any  $\xi \in B^*$  with  $\xi \rho' \in \mathbb{Z}_p^*$ . In particular,  $\tilde{w}_\rho$  is an involution exactly when  $\rho \in B'$ . Finally, the compatibility relation follows easily from the fact that  $\pi_{11}(x, y) = \tau_f(x)$ . □

**Corollary 7.3.** *Every fixed point of  $\tilde{w}_\rho$  lies (via  $\pi_{11}$ ) over a fixed point of  $w_\rho$ . If  $\mathcal{D}_\rho \subseteq \mathbf{SD}_A$  is one of the two residue disks which are preserved by  $w_\rho$ ,  $\tilde{\mathcal{D}}_\rho := \pi_{11}^{-1}(\mathcal{D}_\rho)$  is invariant under  $\tilde{w}_\rho$ .*

*Proof.* These are immediate consequences of  $\pi_{11} \circ \tilde{w}_\rho = w_\rho \circ \pi_{11}$ .  $\square$

**Remark 7.4.** Let  $1_B$  be the multiplicative identity in  $B$ . Then  $w_{1_B}$  is  $w_1|_{SD_A}$  and  $\tilde{w}_{1_B}$  is  $w_3|_{\mathbf{Z}_A}$ .

Recall from Proposition 4.9 that the fixed points of  $w_\rho$  correspond to pairs  $(E, C)$ , where  $E$  has fake CM by  $\mathbb{Z}_p[\pi]$  and  $\ker(\pi) = C$ . The points of  $\mathbf{Z}_A$  which lie over such a fixed point then correspond to pairs,  $(E/H, p^{-1}D/H)$ , where  $H$  and  $D$  are as in the proof of Proposition 7.1. In particular,  $H \subseteq E$  is a non-canonical subgroup of order  $p$ , and  $D \subseteq E$  is cyclic of order  $p^2$  such that  $pD = C$ . Combining these facts with Corollary 7.3 gives us a convenient way to describe (and count) the fixed points of  $\tilde{w}_\rho$ .

**Proposition 7.5.** Let  $(E, C)$  be a fixed point of  $w_\rho$  for some  $\rho \in B^*$ , such that  $\text{End}(\hat{E}) = \mathbb{Z}_p[\pi]$  with  $\ker(\pi) = C$ . If  $\rho \in B'(1 + pjB)$ , there are  $p$  fixed points of  $\tilde{w}_\rho$  lying over  $(E, C)$ , specifically those pairs  $(E/H, p^{-1}D/H)$  with  $\pi(D) = H$ . Otherwise,  $\tilde{w}_\rho$  has no fixed points.

*Proof.* Fix a Woods Hole triple,  $(F, \alpha, C)$ , corresponding to  $(E, C)$ . Then  $E/C$  is equivalent to some pair,  $(F/C, \beta)$ , such that the diagram from the proof of Proposition 4.9 commutes. Note that an explicit isomorphism from  $\rho(E/C)$  to  $E$  is then given by the pair,  $(\sigma, id)$ . In order to determine the  $\tilde{w}_\rho$  fixed points, it will be useful to similarly describe the isomorphism from  $\rho'(E)$  to  $E/C$  which exists by  $\rho \circ w_1 = w_1 \circ \rho'$ . This can be done by replacing  $\rho \circ j$  with  $j \circ \rho'$  in the diagram, and repeating the first isogeny, to obtain the following.

$$\begin{array}{ccccccc} \bar{F} & \xrightarrow{\iota_C} & \overline{F/C} & \xrightarrow{\bar{\sigma}} & \bar{F} & \xrightarrow{\iota_C} & \overline{F/C} \\ \rho' \circ \alpha \downarrow & & \rho \circ \beta \downarrow & & \alpha \downarrow & & \beta \downarrow \\ \hat{A} & \xrightarrow{j} & \hat{A} & \xrightarrow{id} & \hat{A} & \xrightarrow{j} & \hat{A} \end{array}$$

Since  $j^2 = -p$ , this diagram shows that an isomorphism from  $\rho'(E)$  to  $E/C$  is given by the pair,  $(\gamma, id)$ , where  $\gamma = -p^{-1}\iota_C \circ \sigma \circ \iota_C$ .

Now, choose a point lying over  $(E, C)$  by taking  $x = (E, H) = (F, \alpha, H)$  and  $y = (E/C, D/C) = (F/C, \beta, D/C)$ . We must determine when

$$\tilde{w}_\rho(x, y) = (\rho y, \rho' x) = (x, y).$$

Since an isomorphism from  $\rho(E/C)$  to  $E$  is given by  $(\sigma, id)$ , the condition that  $\rho y = x$  is equivalent to  $\sigma(D/C) = H$ . Similarly, the condition that  $\rho' x = y$  is equivalent to  $\gamma(H) = D/C$ . Putting these in terms of  $\pi$ , the first condition is  $\pi(D) = H$  and the second is  $\pi(D) = -(\pi^2/p)(H)$ . By Remark 4.11, these two conditions are equivalent when  $\rho \in B'(1 + pjB)$ , and incompatible otherwise.  $\square$

**Remark 7.6.** If  $(E, C)$  is any point lying over a fixed point of  $w_\rho$  via  $\pi_{11}$ , it is a fake Heegner point in the sense that  $E$  has Fake CM and  $\text{End}(\widehat{E/C})$  is isomorphic to  $\text{End}(\hat{E})$ . In fact, one can show in this case that  $\text{End}(\hat{E}) \cong \mathbb{Z}_p[\lambda]$  for some  $\lambda$  such that  $\ker(\lambda) = C$ .

## 8 Explicit Analysis

In this section, we use Proposition 7.1 and Theorem 3.5 to explicitly compute the reduction of  $\mathbf{Z}_A$  (for  $A/\mathbb{F}_p$  and  $j(A) \neq 0, 1728$ ), in much the same way that the reduction of  $\mathbf{Y}_A$  was computed in the proof of Proposition 5.2. We obtain the equation:

$$X^{p+1} + X^{-(p+1)} = Z^p.$$

Moreover, the residue classes of  $\mathbf{Z}_A$  which have singular reduction on this model are shown to coincide with those regions,  $\tilde{\mathcal{D}}_\rho$ , which were described in Corollary 7.3. From the previous section we know that  $\tilde{\mathcal{D}}_\rho$  is acted on by the involution,  $\tilde{w}_\rho$ , with  $p$  fixed points. In addition, from the explicit equation for  $\tilde{\mathbf{Z}}_A$ , we are able to deduce that  $\tilde{\mathcal{D}}_\rho$  is a connected wide open with one end, and that  $\tilde{\mathcal{D}}_\rho/\tilde{w}_\rho$  is a disk. Putting all of this information together (and a little more), we are able to show in §8.2 that  $\tilde{\mathcal{D}}_\rho$  is a basic wide open whose underlying affinoid reduces to  $y^2 = x^p - x$ .

### 8.1 Reduction of $\mathbf{Z}_A$

Recall that Proposition 7.1 identifies  $\mathbf{Z}_A$  with the subspace of  $\mathbf{C}_A \times \mathbf{C}_A$  defined by  $\tau_f(x) = w_1 \circ \tau_f(y)$ . From this embedding we can obtain an explicit equation for  $\mathbf{Z}_A$ , provided we can derive approximation formulas for  $w_1$  on  $\mathbf{SD}_A$  and  $\tau_f : \mathbf{C}_A \rightarrow \mathbf{SD}_A$ . Such formulas follow readily from Theorem 3.5. However, while the formula in this theorem is given over  $\mathbb{Q}_p \otimes W(\mathbb{F}_{p^2})$ , we will ultimately need to work over a finite base extension. This extension can be generated by fixing a square root of  $\kappa$  in  $\mathbb{C}_p$ , denoted  $\sqrt{\kappa}$  (where  $\kappa$  is as in Theorem 3.5), and a  $\beta \in \mathbb{C}_p$  satisfying

$$\beta^{p^2} \equiv \kappa \pmod{p^{3/2-1/2p^2}}. \quad (4)$$

**Remark 8.1.** *For example, if  $g(x) = x^{p^2} - \sqrt{\kappa}x$ , and  $\gamma$  is a root of  $g(g(x))/g(x)$ , one may take  $\beta = \gamma^{2(p^2-1)}$ . Then, by Lubin-Tate theory, applied to the Lubin-Tate formal group over  $F := \mathbb{Q}_p(\sqrt{\kappa}) \otimes_{\mathbb{Z}_p} W(\mathbb{F}_{p^2})$ , with endomorphism  $g(x)$ ,  $\mathbb{F}(\beta)$  is Galois over  $F$  with Galois group  $C_p \times C_p$ .*

**Proposition 8.2.** *Over  $R := \mathbb{Z}_p[\sqrt{\kappa}, \beta] \otimes W(\mathbb{F}_{p^2})$ , the reduction of  $\mathbf{Z}_A$  has the equation*

$$X^{p+1} + X^{-(p+1)} = Z^p.$$

*Hence, over  $R$ , its reduction is a reduced, connected, affine curve of genus zero with only one branch through each singular point.*

*Proof.* First we derive an approximation for  $\tau_f : \mathbf{C}_A \rightarrow \mathbf{SD}_A$  in terms of the parameter  $t$  from Theorem 3.5. For any  $P_1 \in \mathbf{SD}_A$  and  $P_2 \in \mathbf{C}_A$ , we note that  $P_1 = \tau_f(P_2)$  if and only if  $\pi_f(P_1) = \pi_f(P_2)$ . Thus, an approximation for  $\tau_f$  should follow from approximations for  $\pi_f$  on  $\mathbf{SD}_A$  and  $\mathbf{C}_A$ . Now, we know from [Bu, 3.3] that  $\mathbf{SD}_A$  and  $\mathbf{C}_A$  are the circles described by  $v(t) = 1/2$  and  $v(t) = 1 - 1/2p$ . In particular, we must have  $v(t(P_1)) = 1/2$  and  $v(t(P_2)) =$

$1 - 1/2p$ . Therefore, from Theorem 3.5 we can approximate  $\pi_f$  on  $\mathbf{SD}_A$  and  $\mathbf{C}_A$  as follows.

$$\begin{aligned} s(\pi_f(P_1)) &\equiv t(P_1) && (\text{mod } p) \\ s(\pi_f(P_2)) &\equiv t(P_2) + (\kappa/t(P_2))^p && (\text{mod } p) \end{aligned}$$

Hence an approximation for  $\tau_f : \mathbf{C}_A \rightarrow \mathbf{SD}_A$  is given by

$$t(\tau_f(P)) \equiv t(P) + (\kappa/t(P))^p \pmod{p}.$$

In order to describe the reduction of  $\mathbf{Z}_A$  via Proposition 7.1, we now choose parameters which identify  $\mathbf{C}_A$  and  $\mathbf{SD}_A$  with the unit circle,  $C[1]$ . For such a parameter on  $\mathbf{SD}_A$  we let  $U := t/\sqrt{\kappa}$ , and on  $\mathbf{C}_A$  we let  $X := t/\alpha$  where

$$\alpha = (\beta^{(p^2+1)/2}/\sqrt{\kappa})^{p(2p-1)}$$

(note that  $v(\alpha) = 1 - 1/2p$ ). In terms of these new parameters, the Atkin-Lehner involution is just given by  $w_1^*U = 1/U$ . Also, using the defining congruence for  $\beta$ , the above approximation formula for  $\tau_f$  becomes

$$\tau_f^*U \equiv \alpha X/\sqrt{\kappa} + X^{-p} \pmod{\sqrt{p}}.$$

Now let  $Y$  and  $V$  be analogous parameters on copies of  $\mathbf{C}_A$  and  $\mathbf{SD}_A$ , so that the equation  $\tau_f(P) = w_1(\tau_f(Q))$  on  $\mathbf{C}_A \times \mathbf{C}_A$  (which defined the subspace  $S \cong \mathbf{Z}_A$ ) becomes  $\tau_f^*U = 1/\tau_f^*V$ . Then on  $S$  the parameters  $X$  and  $Y$  satisfy the following congruence relations.

$$(\alpha X/\sqrt{\kappa} + X^{-p})(\alpha Y/\sqrt{\kappa} + Y^{-p}) \equiv 1 \pmod{\sqrt{p}}$$

$$\alpha X^{p+1}/\sqrt{\kappa} + \alpha Y^{p+1}/\sqrt{\kappa} + 1 \equiv X^p Y^p \pmod{\sqrt{p}} \quad (5)$$

Finally, we define a new parameter  $Z$  on  $\mathbf{C}_A \times \mathbf{C}_A$  by  $XY = \beta^{(p-1)/2}Z + 1$ . Then over  $R \otimes \mathbb{Q}_p$ ,  $\mathbf{Z}_A$  is determined by  $|X| \leq 1$  and  $|Z| \leq 1$ . The congruence,

$$X^{p+1} + X^{-(p+1)} \equiv Z^p \pmod{m_R},$$

where  $m_R$  is the maximal ideal of  $R$ , follows from (5).  $\square$

**Proposition 8.3.** *The involutions  $\tilde{w}_\rho$  on  $\mathbf{Z}_A$  reduce to the involutions on  $\bar{\mathbf{Z}}_A$  given by*

$$t_\zeta : (X, Z) \rightarrow (\zeta/X, Z),$$

where  $\zeta$  varies over all  $p+1$ -st roots of unity. The  $\tilde{\mathcal{D}}_\rho^i$  coincide with the singular residue classes of  $\mathbf{Z}_A$ , which are described by  $X^{2p+2} \equiv 1$ .

*Proof.* This basically follows from the compatibility relation in the proof of Corollary 7.2, namely  $\pi_{11} \circ \tilde{w}_\rho = w_\rho \circ \pi_{11}$ . Recall from Proposition 7.1 that

$\pi_{11}(x, y) = \tau_f(x)$  (with notation consistent with that of the previous proposition). So from the proof of the previous proposition, an explicit formula for  $\pi_{11}$  as a map from  $\overline{\mathbf{Z}}_A$  to  $\overline{\mathbf{SD}}_A$  is

$$U = \pi_{11}(X, Z) = X^{-p}.$$

Now, we know from Proposition 4.7 that on  $\overline{\mathbf{SD}}_A$  the involutions  $w_\rho$  reduce to those of the form  $U \rightarrow \zeta/U$  (where  $\zeta$  is any  $(p+1)$ -st root of unity). So fix a  $\rho$  and corresponding  $\zeta$ . Choose any point  $(X_0, Z_0)$  on  $\overline{\mathbf{Z}}_A$ , and let  $(X_1, Z_1) = \tilde{w}_\rho(X_0, Z_0)$ . We can compute both sides of the above compatibility relation as follows.

$$\begin{aligned} w_\rho \circ \pi_{11}(X_0, Z_0) &= w_\rho(X_0^{-p}) = \zeta X_0^p \\ \pi_{11} \circ \tilde{w}_\rho(X_0, Z_0) &= \pi_{11}(X_1, Z_1) = X_1^{-p} \end{aligned}$$

Since  $\zeta = \zeta^{-p}$ , we must have  $X_1 = \zeta/X_0$  and subsequently  $Z_1 = Z_0$ . In other words, we have shown that on  $\overline{\mathbf{Z}}_A$  we have  $\tilde{w}_\rho(X, Z) = (\zeta/X, Z)$ .

Keeping the same notation, the points of  $\overline{\mathbf{SD}}_A$  which are fixed by  $w_\rho$  are the two described by  $U^2 = \zeta$ , and by definition  $\tilde{\mathcal{D}}_\rho^1$  and  $\tilde{\mathcal{D}}_\rho^2$  are  $\pi_{11}^{-1}$  of the corresponding residue classes. Since  $\pi_{11} : \overline{\mathbf{Z}}_A \rightarrow \overline{\mathbf{SD}}_A$  is given by  $U = X^{-p}$ , this is equivalent to saying that  $\tilde{\mathcal{D}}_\rho^i$  are the classes of  $\overline{\mathbf{Z}}_A$  described by  $X^2 \equiv \zeta$ . Letting  $\zeta$  vary over all  $p+1$ -st roots of unity we obtain all the residue classes described by  $X^{2p+2} \equiv 1$ , and these are easily verified to be the singular ones.  $\square$

**Proposition 8.4.** *For any  $\rho \in B'$ , the residue classes of the affinoid quotient,  $\mathbf{Z}_A/\tilde{w}_\rho$ , which are the images of the  $\tilde{\mathcal{D}}_\rho^i$  are disks over  $\mathbb{Z}_p[\sqrt{\kappa}, \beta] \otimes W(\mathbb{F}_{p^2})$ .*

*Proof.* Let  $\zeta$  be the  $p+1$ -st root of unity such that  $\tilde{w}_\rho$  reduces to  $t_\zeta$  on  $\overline{\mathbf{Z}}_A$ . Let  $f_\zeta(x)$  be the unique polynomial of degree  $p+1$  such that

$$f_\zeta(X + \zeta/X) = X^{p+1} + X^{-(p+1)}.$$

Then  $f_\zeta(x) = z^p$  is an equation for the reduction of  $\mathbf{Z}_A/\tilde{w}_\rho$ . Moreover,

$$f'_\zeta(X + \zeta/X) = \frac{X^{2p+2} - 1}{X^p(X^2 - \zeta)},$$

and the right hand side doesn't vanish at  $\epsilon$  if  $\epsilon^2 = \zeta$ . Thus  $f'_\zeta(2\epsilon) \not\equiv 0 \pmod{p}$ , and the two residue classes of  $\mathbf{Z}_A/\tilde{w}_\rho$  described by  $X = \pm\epsilon$  are disks.  $\square$

From Theorem 2.29 and Proposition 2.31, we now conclude that (over a suitable field extension)  $\tilde{\mathcal{D}}_\rho^i$  is a connected wide open with one end. Furthermore, using Theorem 2.48 and the fact that there are  $p$  branch points in the degree 2 quotient of  $\tilde{\mathcal{D}}_\rho^i$  by  $\tilde{w}_\rho$ , we compute the genus of  $\tilde{\mathcal{D}}_\rho^i$  to be  $(p-1)/2$ . To summarize, we have the following corollary.

**Corollary 8.5.** *Let  $L$  be a complete stable subfield of  $\mathbb{C}_p$  containing  $R$ , over which the fixed points of  $\tilde{w}_\rho$  are defined. Over  $L$ , the rigid spaces  $\tilde{\mathcal{D}}_\rho^i$ ,  $i = 1$  or  $2$ , are connected wide opens with one end of genus  $(p-1)/2$ .*

## 8.2 The New Components

We now show that over a suitable base extension the  $2(p+1)$  residue classes,  $\tilde{\mathcal{D}}_\rho^i \subseteq \mathbf{Z}_A$ , are basic wide opens, and we compute the reductions of their underlying affinoids. The main idea will be to construct an automorphism of order  $p$  on each  $\tilde{\mathcal{D}}_\rho^i$  which transitively permutes the  $p$  fixed points of the involution  $\tilde{w}_\rho$ . This induces an automorphism on the quotient,  $\tilde{\mathcal{D}}_\rho^i/\tilde{w}_\rho$ , a disk by Corollary 8.5, which then must be conjugate to a translation.

First we define automorphisms of order  $p$  on the disk,  $\tau_f^{-1}(\mathcal{D}) \subseteq \mathbf{C}_A$ , where  $\tau_f : \mathbf{C}_A \rightarrow \mathbf{SD}_A$  is as in Section 3, and  $\mathcal{D}$  is either of the two residue disks of  $\mathbf{SD}_A$  fixed by  $w_\rho$ . Recall that points of  $\mathbf{SD}_A$  correspond to pairs  $(E, C)$  where  $h(E) = 1/2$  and  $C$  is canonical. One of the key facts which we use in our construction is that over the residue disk  $\mathcal{D}$  one can analytically choose a generator up to sign for each of these canonical subgroups. This amounts to choosing a section  $\sigma$  of the forgetful map from  $X_1(p)$  to  $X_0(p)$  over  $\mathcal{D}$ ,

$$\sigma : (E, K_1(E)) \mapsto (E, P_\sigma(E))$$

where  $P_\sigma(E)$  is a pair consisting of a generator of  $K_1(E)$  and its inverse. Such a section exists because this map is an étale map of annuli over  $\mathbf{SD}_A$  (over any extension of  $\mathbb{Q}_p$  whose ramification index is divisible by  $2(p-1)$ ). In fact, the group  $(\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$  acts simply transitively on the set of the sections over  $\mathcal{D}$ . Once  $\sigma$  is chosen, automorphisms of  $\tau_f^{-1}(\mathcal{D})/\mathcal{D}$  can be constructed by looking closely at the Weil pairing. In particular, we have the following lemma.

**Lemma 8.6.** *For any  $\zeta \in \mu_p^*$  and  $\sigma$  (as above), we can define an analytic automorphism of  $\tau_f^{-1}(\mathcal{D})/\mathcal{D}$  by  $S_{\sigma, \zeta}(E, H) = (E, \langle R \rangle)$ , where  $R \in E[p]$  is chosen so that  $e_p(P, R) = \zeta$  and  $R - P \in H$  for some  $P \in P_\sigma(E)$ . Moreover, we have:*

$$(i) S_{\sigma, \zeta}^i(E, H) = (E, \langle R + (i-1)P \rangle) \text{ for } i \in \mathbb{Z}.$$

(ii)  $f_{\sigma, \zeta} : \mathbb{Z}/p\mathbb{Z} \rightarrow \text{Aut}_{an}(\tau_f^{-1}(\mathcal{D})/\mathcal{D})$ , defined by  $f_{\sigma, \zeta}(i) = S_{\sigma, \zeta}^i$  for  $i \neq 0$  and the identity otherwise, is an injective homomorphism.

$$(iii) S_{a\sigma, \zeta^b} = S_{\sigma, \zeta}^{a^2/b} \text{ for any } a, b \in (\mathbb{Z}/p\mathbb{Z})^*.$$

$$(iv) S_{\sigma, \zeta}^\tau = S_{\sigma^\tau, \zeta^\tau} \text{ for any } \tau \in \text{Aut}_{cont}(\mathbb{C}_p) \text{ which preserves } \mathcal{D}.$$

*Proof.* Fix  $\sigma$  and  $\zeta \in \mu_p^*$ . For a given pair  $(E, H)$  and choice of  $P \in P_\sigma(E)$ , there is a unique  $R \in E[p]$  which satisfies the two conditions. Note also that reversing the sign of  $P$  just reverses the sign of  $R$ . Since  $\langle R \rangle = \langle -R \rangle$  is neither  $H$  nor the canonical subgroup, it follows that  $S_{\sigma, \zeta}$  is at least a well-defined automorphism of  $\tau_f^{-1}(\mathcal{D})/\mathcal{D}$  with no fixed points.

Now, fix  $P \in P_\sigma(E)$ . It is easy to verify (i) by induction, and then (ii) follows immediately. To prove (iii), we note that by definition  $S_{a\sigma, \zeta^b}(E, H)$  is the pair  $(E, \langle Q \rangle)$  where  $e_p(aP, Q) = \zeta^b$  and  $Q - aP \in H$ . Well, a simple Weil pairing calculation shows that  $Q$  is just  $aP + (b/a)(R - P)$ . So we can verify (iii) by checking that

$$e_p(aP + (b/a)(R - P), R + (a^2/b - 1)P) = 1.$$

Finally, property (iv) follows from Galois properties of the Weil pairing and the fact that  $\mathcal{D}$  is connected.  $\square$

**Proposition 8.7.** *Let  $L$  be a finite extension of  $\mathbb{Q}_p(\sqrt{\kappa}, \beta)$  in  $\mathbb{C}_p$  where  $\sqrt{\kappa}$  and  $\beta$  are as in Equation (4), over which the fixed points of  $\tilde{w}_\rho$  are defined. Then  $\tilde{\mathcal{D}}_\rho^i$  is a basic wide open over a quadratic extension of  $L$ , whose underlying affinoid has good reduction, which can be described by the equation*

$$y^2 = x^p - x.$$

*Proof.* As usual, let  $\mathcal{D}$  be either of the residue disks of  $\mathbf{SD}_A$  fixed by the involution  $w_\rho$ , and  $\tilde{\mathcal{D}}$  the wide open lying over  $\mathcal{D}$  via  $\pi_{11}$ . Then the embedding of  $\mathbf{Z}_A = \pi_{11}^{-1}(\mathbf{SD}_A)$  into  $\mathbf{C}_A \times \mathbf{C}_A$  embeds  $\tilde{\mathcal{D}}$  into  $\tau_f^{-1}(\mathcal{D}) \times \rho' \tau_f^{-1}(\mathcal{D})$ . Therefore, by the previous lemma we can lift any automorphism  $S := S_{\sigma, \zeta}$  on  $\tau_f^{-1}(\mathcal{D})$  (for a fixed  $\sigma$  and  $\zeta$ ) to an automorphism  $\tilde{S} := \tilde{S}_{\sigma, \zeta}$  of  $\tilde{\mathcal{D}}$  by taking

$$\tilde{S}(x, y) = (S(x), \rho' S(\rho y)).$$

One easily checks that  $\tilde{S}$  also has order  $p$ , since

$$\tilde{S}^i(x, y) = (S^i(x), \rho' S^i(\rho y)).$$

Furthermore,  $\tilde{S}$  commutes with  $\tilde{w}_\rho$ .

$$\begin{aligned} \tilde{S}\tilde{w}_\rho(x, y) &= \tilde{S}(\rho y, \rho' x) \\ &= (S(\rho y), \rho' S(\rho' x)) = (S(\rho y), \rho' S(x)) \\ \tilde{w}_\rho\tilde{S}(x, y) &= \tilde{w}_\rho(S(x), \rho' S(\rho y)) \\ &= (\rho\rho' S(\rho y), \rho' S(x)) = (S(\rho y), \rho' S(x)) \end{aligned}$$

It follows that  $\tilde{S}$  passes to an automorphism of  $\tilde{\mathcal{D}}/\tilde{w}_\rho$  with order  $p$  and no fixed points, and which acts transitively on the images of the  $p$  fixed points of  $\tilde{w}_\rho$ . This is the key idea in the proof of the proposition.

To finish the argument, recall from Corollary 8.5 that  $\tilde{\mathcal{D}}$  is a connected wide open with one end. The involution  $\tilde{w}_\rho$  acts on it with  $p$  fixed points, and the quotient space, say  $U := \tilde{\mathcal{D}}/\tilde{w}_\rho$ , is a disk by Proposition 8.4. It follows that, over a quadratic extension of  $L$ ,  $\tilde{\mathcal{D}}$  can be described by the equation

$$y_0^2 = (x_0 - \alpha_1) \cdots (x_0 - \alpha_p),$$

where  $x_0$  is a parameter for  $U$  and the  $\alpha_i$ 's are the  $x_0$  coordinates of the  $p$  fixed points. Without loss of generality, we choose  $x_0$  so that  $U$  is identified with the disk,  $v(x_0) > 0$ . Because  $\tilde{S}$  passes to an automorphism of a disk of order  $p$  and no fixed points, it must reduce to a translation, in the sense that there exists an  $a \in R_p$  with  $v(a) > 0$  such that for all  $x_0 \in U$  we have

$$v(\tilde{S}(x_0) - (x_0 + a)) > v(a).$$

Therefore, after possible reordering, the  $x_0$  coordinates of the fixed points must satisfy

$$\beta_i := \frac{\alpha_i - \alpha_1}{a} \equiv i \pmod{m_p}.$$

So if we make the change of variables,  $x = (x_0 - \alpha_1)/a$  and  $y = y_0/a^{p/2}$ , we identify  $\tilde{\mathcal{D}}$  with the wide-open

$$y^2 = (x - \beta_1)(x - \beta_2) \cdots (x - \beta_p) \quad v(x) > -v(a)$$

which reduces as claimed.  $\square$

**Remark 8.8.** *The results of Section 8 were proven only for  $A/\mathbb{F}_p$  with  $j(A) \neq 0, 1728$ , but similar results now follow for any other supersingular  $A'$  using Proposition 4.2. Since  $\mathbf{Z}_{A'}$  is an étale quotient of  $\mathbf{Z}_A$  of degree  $i(A)$ ,  $\overline{\mathbf{Z}}_{A'}$  is a genus 0 curve with  $2(p+1)/i(A')$  singular points, corresponding to basic wide opens which are isomorphic to those described in Proposition 8.7. Note, however, that one might need to replace the field  $L$  from Corollary 8.5 by a finite unramified extension in order to define the surjection from  $W_A(p^3)$  onto  $W_{A'}(p^3)$  and describe the underlying affinoids. In general, the reduction of the bridging component has the equation,*

$$X^{(p+1)/i(A)} + X^{-(p+1)/i(A)} = Z^p.$$

**Lemma 8.9.** *The Hecke correspondence  $T_\ell$  takes a divisor supported on  $A$  to a divisor supported on  $A$ , if  $A$  is the union of the minimal underlying affinoids of the bridging components for all primes  $\ell$ .*

*Proof.* A point,  $(E, C)$ , lies on the minimal underlying affinoid of a bridging component if and only if  $C$  is cyclic of order  $p^3$  and  $pC/p^2C$  is self dual. If  $f : E \rightarrow F$  is an isogeny such that  $\ker(f) \cap C = 0$ , the same is true for  $(F, f(C))$ .  $\square$

**Remark 8.10.** *The same statement, where  $A$  is the union of the minimal underlying affinoids corresponding to new components for all primes  $\ell \neq p$  will follow from the results of [CMc, §8] where higher tame level is treated.*

## 9 Stable Reduction of $X_0(p^3)$

In this section we give the stable covering of  $X_0(p^3)$ . In particular we give a covering by *basic* wide opens, whose intersections are annuli as in Proposition 2.34. Some of these wide opens, namely the  $W_{ab}^\pm$ , have already been defined in Section 3. They cover the ordinary locus, and will be shown to be basic with the  $\mathbf{X}_{ab}^\pm$  as underlying affinoids. From our analysis of  $\mathbf{Z}_A$  in Section 8, we now know that  $W_A(p^3)$  is not a basic wide open. So our next priority is to specify some new wide open subspaces which cover each  $W_A(p^3)$  and which can ultimately be shown to be basic.

So suppose now that  $A$  is any supersingular elliptic curve mod  $p$  (no restrictions). Identify  $W_{A^\sigma}(p)$  (where  $\sigma$  is the Frobenius automorphism) with the annulus,  $A(p^{-i(A)}, 1)$ , as explained in Section 3.1. Then we can define three subspaces of  $W_A(p^3)$  in the following way.

$$\begin{aligned} V_1(A) &:= \pi_{11}^{-1}A(p^{-i(A)}, p^{-i(A)/2}) \\ V_2(A) &:= \pi_{11}^{-1}A(p^{-i(A)/2}, 1) \\ U(A) &:= \pi_{11}^{-1}A(p^{-pi(A)/(p+1)}, p^{-i(A)/(p+1)}) \end{aligned}$$

First we want to show that these subspaces are wide opens (over  $\mathbb{C}_p$ ). Since  $V_1(A)$  is a union of residue classes of the affinoid,

$$\pi_{11}^{-1} \left( X_{01} \cup A(p^{-i(A)}, p^{-i(A)/2}) \right),$$

and since it is connected, it is in fact one residue class and therefore a wide open by Theorem 2.29. The same argument applies to  $V_2(A)$  and  $U(A)$ , the latter being a residue class of

$$\pi_{11}^{-1}A[p^{-pi(A)/(p+1)}, p^{-i(A)/(p+1)}].$$

**Remark 9.1.** *The points of  $A(p^{-pi(A)/(p+1)}, p^{-i(A)/(p+1)})$  are pairs,  $(E, C)$ , where  $C$  is the canonical subgroup of  $E$  and  $E[p]/C$  is the canonical subgroup of  $E/C$ .*

Two of these supersingular wide opens will in fact be shown to be basic. More specifically,  $V_1(A)$  is a wide open neighborhood of the affinoid,  $\mathbf{E}_{1A} := \pi_{11}^{-1}C[p^{-pi(A)/(p+1)}]$  (which will be shown to be an underlying affinoid with good reduction). Points of  $\mathbf{E}_{1A}$  are pairs,  $(E, C)$ , such that  $E/p^2C$  is too-supersingular. Alternatively,  $\mathbf{E}_{1A}$  can be described as  $\pi_f^{-1}\mathbf{Y}_A$ , which is a key point because it implies that  $\mathbf{E}_{1A}$  is “nontrivial.” Similarly,  $V_2(A)$  is a neighborhood of  $\mathbf{E}_{2A} := \pi_{11}^{-1}C[p^{-i(A)/(p+1)}]$ . Points of  $\mathbf{E}_{2A}$  are pairs  $(E, C)$  with  $E/pC$  too-supersingular, and  $\mathbf{E}_{2A}$  maps onto  $\mathbf{Y}_{A^\sigma}$  via  $\pi_\nu$ .  $U(A)$  is not basic, because its underlying affinoid,  $\mathbf{Z}_A$ , has the  $\tilde{\mathcal{D}}_\rho^i$  as (bad) residue classes. However, the  $\tilde{\mathcal{D}}_\rho^i$  were shown to be basic in Proposition 8.7. So this problem can essentially be solved by removing the underlying affinoids of the  $\tilde{\mathcal{D}}_\rho^i$  from  $U(A)$  (obtaining a basic wide open) and then including the  $\tilde{\mathcal{D}}_\rho^i$  in the overall covering. To be more precise, let  $\mathcal{S}(A)$  denote the set of singular residue classes of  $\mathbf{Z}_A$ , and for each  $S \in \mathcal{S}(A)$  let  $\mathbf{X}_S$  be the underlying affinoid of  $S$ . Let  $\hat{U}(A)$  denote the wide open given by

$$\hat{U}(A) := U(A) \setminus \bigcup_{S \in \mathcal{S}(A)} \mathbf{X}_S.$$

**Theorem 9.2.** *Let  $p \geq 13$  be a prime. The covering  $\mathcal{C}_0(p^3)$  of  $X_0(p^3)$ , which is made up of*

$$\{ W_{ab}^\pm \mid a, b \geq 0, a + b = 3 \}$$

and the union over all supersingular curves  $A$  of

$$\{ V_1(A), V_2(A), \hat{U}(A) \} \cup \mathcal{S}(A),$$

is stable (over  $\mathbb{C}_p$ ).

*Proof.* We know that the elements of  $\mathcal{C}_0(p^3)$  are wide opens, and that  $(S, \mathbf{X}_S)$  is a basic wide open pair for each  $S \in \mathcal{S}(A)$ . It is also easy to verify that condition (ii) of Proposition 2.34 holds, by simply listing for each wide open the other members of the covering which intersect it nontrivially. In particular, the  $W_{ab}^\pm$  are disjoint from each other, and each  $W_{ab}^\pm$  intersects  $W_A(p^3)$  only at  $V_2(A)$  when  $a > b$  and only at  $V_1(A)$  otherwise. Similarly, while  $V_1(A)$ ,  $V_2(A)$ , and the residue classes  $S \in \mathcal{S}(A)$  are pairwise disjoint, each of these wide opens intersects  $\hat{U}(A)$  nontrivially. This completely describes all adjacency relations of wide opens in the covering, and it follows immediately that every triple intersection is empty. The bulk of what we still have to show is that whenever two wide opens in the cover do intersect, the intersection is the disjoint union of annuli. Then we have to show that each wide open is basic, with an underlying affinoid that has good reduction.

We start by showing that  $U_{ab}^\pm(A) := W_{ab}^\pm \cap W_A(p^3)$  is a wide open annulus in all cases. For  $U_{30}$  and  $U_{21}^\pm$  it suffices to consider the map  $\pi_{02}$  from  $X_0(p^3)$  to  $X_0(p)$ . The restriction of  $\pi_{02}$  to  $U_{30}$  is an isomorphism onto the annulus,  $B := A(p^{-\frac{i(A)}{p(p+1)}}, 1) \cong A(1, p^{\frac{i(A)}{p(p+1)}})$  (considered as a subspace of  $W_A(p)$ , which has been identified with  $A(p^{-i(A)}, 1)$  as in Section 3.1). So  $U_{30}$  is an annulus right away.  $U_{21}^+$  and  $U_{21}^-$  also map onto  $B$  via  $\pi_{02}$ , but each with degree  $(p-1)/2$ . To see that  $U_{21}^\pm$  is at least connected, we look at how  $\pi_{02}$  reduces when restricted to a map between the affinoid regions,  $\mathbf{X}_{21}^\pm$  and  $\mathbf{X}_{10}$ . The latter is an isomorphic copy of the ordinary locus of  $X(1)$ , and by [C3, §1, pg. 5] the reduction of  $\mathbf{X}_{21}^\pm$  is isomorphic to the ordinary locus of  $Ig(p)$ . Furthermore, via these identifications  $\pi_{02}$  reduces to the forgetful map from  $Ig(p)$  to  $X(1)$ , which is totally ramified at the supersingular points. This implies that one of the ends of  $B$  totally ramifies in the restriction of  $\pi_{02}$  to  $U_{21}^\pm$ . Hence  $U_{21}^\pm$  must be connected. Now it follows directly from Theorem 2.6 that  $U_{21}^\pm$  is an annulus. Similar arguments can be made for  $U_{12}^\pm$  and  $U_{03}$  using  $\pi_{20}$ . Alternatively one can use the fact that the Atkin-Lehner involution,  $w_3$ , switches  $W_{ab}$  with  $W_{ba}$  and  $W_A(p^3)$  with  $W_{A^\sigma}(p^3)$ . Note that from this argument we also deduce that each  $(W_{ab}^\pm, \mathbf{X}_{ab}^\pm)$  is a basic wide open pair.

Among the remaining intersections of wide opens in the covering, we also have  $S \cap \hat{U}(A)$  for each  $S \in \mathcal{S}(A)$ . It is immediate, however, that this is an annulus, since  $S$  is a basic wide open with one end and by definition  $S \cap \hat{U}(A)$  is the complement in  $S$  of its underlying affinoid  $\mathbf{X}_S$ . So all that remains to be proven is that  $V_i(A) \cap \hat{U}(A)$  is the disjoint union of annuli (in fact, one annulus), and that  $(V_1(A), \mathbf{E}_{1A})$ ,  $(V_2(A), \mathbf{E}_{2A})$ , and  $(\hat{U}(A), \mathbf{Z}_A)$  are basic wide open pairs. Essentially, this comes down to a genus computation and Proposition 2.34.

First shrink each  $\hat{U}(A)$  to a basic wide open neighborhood  $\hat{U}'(A)$  of  $\mathbf{Z}_A$ , and call the resulting covering  $\mathcal{C}_1(p^3)$ . Although we do not know that  $\mathcal{C}_1$  is

semi-stable (and in fact it isn't), Proposition 2.34 can still be applied as the wide opens in the covering intersect properly in the disjoint union of annuli. Moreover, we know that the intersection of  $\hat{U}'(A)$  with  $V_i(A)$  is just one annulus, because  $\bar{\mathbf{Z}}_A$  has only two points at infinity (see Theorem 2.29). So the Betti number of the graph associated to  $\mathcal{C}_1(p^3)$  is exactly  $5(s_p - 1)$ , where  $s_p$  is the number of supersingular  $j$ -invariants (mod  $p$ ). To apply Proposition 2.34, we need to know the genera of the wide opens in  $\mathcal{C}_1(p^3)$ . The genus of  $W_{ab}$  is 0 when  $ab = 0$  and  $g(Ig(p))$  otherwise by [C3, §1]. The genus of  $\hat{U}'(A)$  is 0 and the genus of each  $S \in \mathcal{S}(A)$  is  $(p - 1)/2$ , by Proposition 8.2 and Corollary 8.5. The only genera that aren't immediately available are those of  $V_1(A)$  and  $V_2(A)$ . We can, however, provide a lower bound for these genera. Recall that  $\mathbf{E}_{1A}$  maps onto  $\mathbf{Y}_A$  via  $\pi_f$ , and  $\mathbf{E}_{2A}$  maps onto  $\mathbf{Y}_{A^\sigma}$  via  $\pi_\nu$ . So by a Riemann-Hurwitz argument we know that  $g(V_i(A)) \geq g(\bar{\mathbf{Y}}_A)$  (which we know from Corollary 5.4).

We now compute a lower bound for the genus of  $X_0(p^3)$ , using the above and Proposition 2.34. For brevity we only discuss the case  $p = 12k + 5$ . Then  $s_p = k + 1$  and from [I, p. 103] we have  $g(Ig(p)) = 3k^2 - k$ . There are  $k$  supersingular regions with  $j(A) \neq 0, 1728$ , each of which contributes: two wide opens,  $V_1(A)$  and  $V_2(A)$ , of genus at least  $g(\mathbf{Y}_A) = 6k + 2$ , and  $24k + 12$  residue classes  $S \in \mathcal{S}(A)$  with genus  $6k + 2$ . In addition, we have one supersingular region corresponding to  $j(A) = 0$  which contributes: two wide opens,  $V_1(A)$  and  $V_2(A)$ , of genus at least  $g(\mathbf{Y}_A) = 2k$ , and  $8k + 4$  residue classes  $S \in \mathcal{S}(A)$  of genus  $6k + 2$ . Summing up the Betti number and genera as in Proposition 2.34 we have

$$\begin{aligned} g(X_0(p^3)) &\leq 5k + 4(3k^2 - k) + 2(2k) + (8k + 4)(6k + 2) + \\ &\quad k[2(6k + 2) + (24k + 12)(6k + 2)] \\ &\leq 144k^3 + 192k^2 + 73k + 8. \end{aligned}$$

This is now easily verified to be the *actual* genus of  $X_0(p^3)$  using the well-known genus formula (see [Sh, Prop. 1.40, 1.43]). Thus the above inequalities are actually equalities. Furthermore, as  $g(V_i(A)) \geq g(\bar{\mathbf{E}}_{iA}) \geq g(\bar{\mathbf{Y}}_A)$ , Lemma 2.43 implies that  $V_1(A)$  and  $V_2(A)$  are basic wide opens such that  $\mathbf{E}_{1A}$  and  $\mathbf{E}_{2A}$  are Zariski subaffinoids of the underlying affinoids. Then, since the reductions of these affinoids each have at least four points at infinity, and since  $V_i(A)$  has only four ends, it follows that  $\mathbf{E}_{1A}$  and  $\mathbf{E}_{2A}$  are the underlying affinoids (with good reduction). Therefore  $V_i(A) \cap \hat{U}(A)$  must be an annulus, and we have shown that  $\mathcal{C}_0(p^3)$  is a stable covering.  $\square$

**Remark 9.3.** *Since  $\mathbf{E}_{1A} = \pi_f^{-1}(\mathbf{Y}_A)$ , and since  $\mathbf{E}_{1A}$  has good reduction with  $g(\bar{\mathbf{E}}_{1A}) = g(\bar{\mathbf{Y}}_A)$ , it follows that*

$$\pi_f : \bar{\mathbf{E}}_{1A} \rightarrow \bar{\mathbf{Y}}_A$$

*is purely inseparable and factors as Frobenius followed by an isomorphism. Hence,  $\bar{\mathbf{E}}_{1A} \cong \bar{\mathbf{Y}}_A^\sigma$ , and similarly  $\bar{\mathbf{E}}_{2A} \cong \bar{\mathbf{Y}}_{A^\sigma}^\sigma$ .*

## 9.1 Graphs and Intersection Data

From Theorem 9.2, it is now straightforward to generate graphs for the stable reduction of  $X_0(p^3)$  according to the four classes of  $p \pmod{12}$ , and we include these graphs below in Figures 2-5. To make the graphs more understandable, a brief description of how the various components are organized and labelled may be in order. First of all, recall from Section 3.2 that there are six ordinary components in every case, namely those corresponding to  $\mathbf{X}_{30}$ ,  $\mathbf{X}_{21}^\pm$ ,  $\mathbf{X}_{12}^\pm$ , and  $\mathbf{X}_{03}$ . These are always presented as vertical components and labelled explicitly with their genera. In addition to the six ordinary components, we have one connected, acyclic configuration of components for each supersingular elliptic curve  $A$ . This configuration is always presented as a horizontal chain of three components, corresponding to  $\mathbf{E}_{2A}$ ,  $\mathbf{Z}_A$ , and  $\mathbf{E}_{1A}$  (in that order), along with a number of unmarked vertical components intersecting the middle component. We explicitly label the genera of the reductions of  $\mathbf{E}_{1A}$  and  $\mathbf{E}_{2A}$ , but not the central “bridging component” as it always has genus 0. Below the central horizontal component, we list the number of copies of  $y^2 = x^p - x$  which intersect it, as well as the genus of each copy. Finally, we point out for clarification that the components corresponding to  $\mathbf{X}_{30}$  and  $\mathbf{X}_{21}^\pm$  meet each supersingular region in exactly one point in the reduction of  $\mathbf{E}_{2A}$ , while the same can be said for the other three ordinary components and  $\mathbf{E}_{1A}$ . In particular, one is reading the graph properly if the Betti number (equivalently the toric rank of the Jacobian) appears to be  $5(ss - 1)$ , where  $ss$  is the number of supersingular  $j$ -invariants. This fact generalizes, as we show in the following theorem.

**Theorem 9.4.** *The toric rank of  $J_0(Np^n)$  for  $(N, p) = 1$  and  $n \geq 0$  is given by  $(s(N) - 1)(2n - 1)$ , where  $s(N)$  is the number of supersingular points on  $X_0(N) \pmod{p}$ .*

*Proof.* For  $N = 1$  and  $n \leq 1$  this follows from [DR, §VI.6]. After inverting isogenies we have the following exact sequence.

$$0 \rightarrow J_0(p^{n-1}) \rightarrow J_0(p^n) \times J_0(p^n) \rightarrow J_0(p^{n+1}) \rightarrow J_0(p^{n+1})^{\text{new}} \rightarrow 0$$

Now, it follows from Theorem 14.7.2 of [KM] that  $J_0(p^n)^{\text{new}}$  has potential good reduction for  $n > 1$ . Hence, by induction, the theorem is true for all  $J_0(p^n)$ . The result for more general  $N$  follows from essentially the same argument.  $\square$

To go along with the stable reduction graphs, we also include the intersection multiplicities below in Table 1. These numbers have been obtained via a rigid-analytic reformulation which bears mentioning. In particular, suppose that  $X$  and  $Y$  are components of a curve with semi-stable reduction over some extension  $K/\mathbb{Q}_p$ , and that they intersect in an ordinary double point  $P$ . Then  $R(P)$  is an annulus (by Proposition 2.10), say with width  $w(P)$ . In this case, the intersection multiplicity of  $X$  and  $Y$  at  $P$  can be found by

$$M_K(P) = e_p(K) \cdot w(P).$$

Note that while intersection multiplicity depends on  $K$ , the width makes sense even over  $\mathbb{C}_p$ , which in some sense makes width a more natural invariant from the purely geometric perspective.

Now, for our calculations on  $X_0(p^3)$ , we take  $e_p(K) = p^2(p^2 - 1)$ , as this is the ramification index over  $\mathbb{Q}_p$  for the field of Krir (see [CMc, §5] for more details). First we address those singular points where  $\mathbf{E}_{iA}$  meets either an ordinary component or the bridging component. Reduction inverse of any such singular point is an annulus in the supersingular locus which surjects via the forgetful map onto some sub-annulus of  $W_A(p)$ . Using Hasse invariant and canonical subgroup considerations, we can determine this sub-annulus and in particular its width. Then we apply Proposition 2.2. For example, the ordinary component corresponding to  $\mathbf{X}_{30}$  intersects the one corresponding to (each)  $\mathbf{E}_{2A}$  in a unique singular point. As we saw in the proof of Theorem 9.2, the corresponding annulus maps via  $\pi_{02}$  (the forgetful map) onto the sub-annulus of  $W_A(p)$  described by

$$0 < v(x_A) < \frac{i(A)}{p(p+1)}$$

with degree 1. The ordinary components corresponding to  $\mathbf{X}_{21}^\pm$  also meet the reduction of  $\mathbf{E}_{2A}$  in exactly one singular point (each). The corresponding two annuli surject onto this same sub-annulus, but with degree  $(p-1)/2$ . Using this line of reasoning, we arrive at most of the data in Table 1. Note that any two components intersect in at most one point, and so we may designate a singular point in the stable reduction unambiguously by listing a pair of intersecting components.

The only intersection multiplicities which do *not* follow readily from the above reasoning come from singular points where a copy of  $y^2 = x^p - x$  (denoted  $\mathbf{X}_S$  for  $S \in \mathcal{S}(A)$  as in the theorem) intersects a bridging component. At such a singular point, the corresponding annulus maps via  $\pi_{11}$  onto an annulus which is the complement of an affinoid disk inside a residue disk of  $\mathbf{SD}_A$ . Unfortunately, it is not at all clear what the width of this image annulus is. We have some theoretical evidence and some computational evidence (for example, see [M1, Remark, pg. 27]) which suggests that the width of the original annulus, i.e. the annulus of intersection, is  $1/(4p^2)$ . Therefore, we have included this in Table 1 with an asterisk to indicate that it is our current *best guess*.

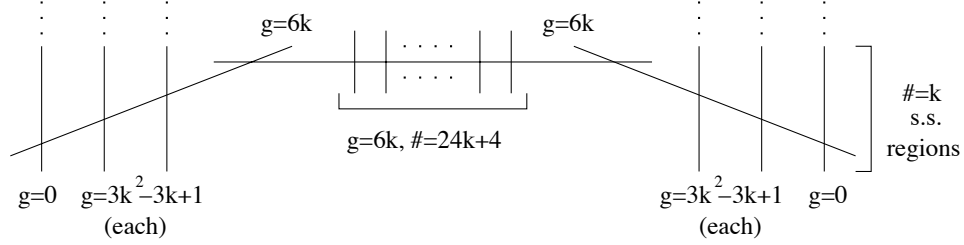


Figure 2: Graph of  $X_0(p^3)$  when  $p = 12k + 1$

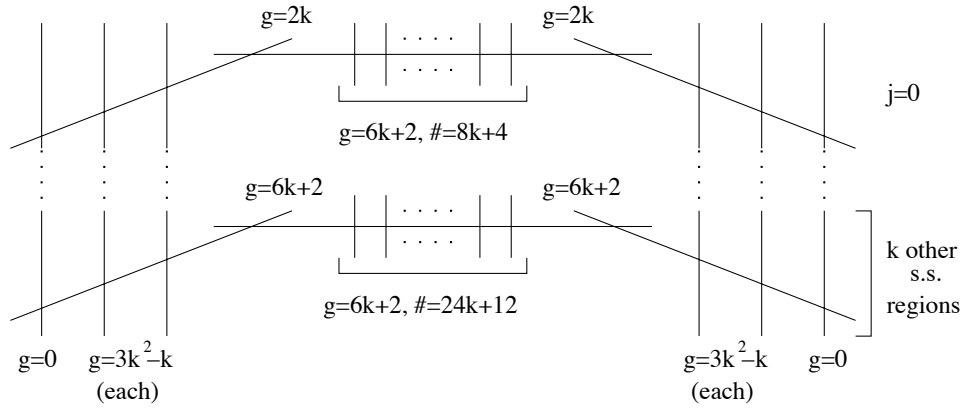


Figure 3: Graph of  $X_0(p^3)$  when  $p = 12k + 5$

$P$	$(X_{30}, E_{2A}), (X_{03}, E_{1A})$	$(X_{21}^{\pm}, E_{2A}), (X_{12}^{\pm}, E_{1A})$	$(Z_A, E_{2A}), (Z_A, E_{1A})$	$(X_S, Z_A)$
$w(P)$	$\frac{i(A)}{p(p+1)}$	$\frac{2 \cdot i(A)}{p(p^2-1)}$	$\frac{(p-1) \cdot i(A)}{2p^2(p+1)}$	$\frac{1}{4p^2} *$
$M_K(P)$	$p(p-1) \cdot i(A)$	$2p \cdot i(A)$	$\frac{(p-1)^2 \cdot i(A)}{2}$	$\frac{p^2-1}{4} *$

Table 1: Intersection Multiplicity Data for  $X_0(p^3)$

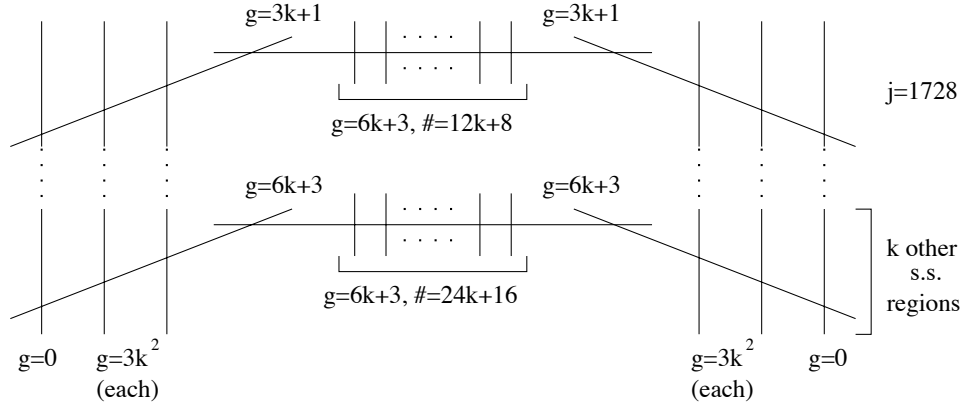


Figure 4: Graph of  $X_0(p^3)$  when  $p = 12k + 7$

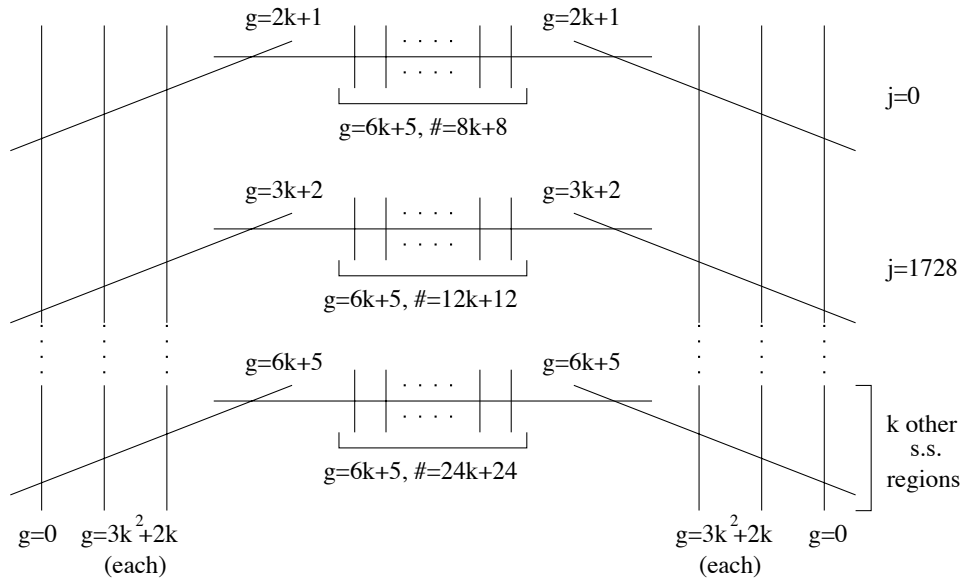


Figure 5: Graph of  $X_0(p^3)$  when  $p = 12k + 11$

## A Riemann Existence Theorem

The  $p$ -adic Riemann Existence Theorem is well known, but not apparently in the literature.<sup>20</sup> Here we recall and adapt the proof of the existence of global

<sup>20</sup>possibly because it follows from the same lines of reasoning as are used in “the” complex case (see [Sp] for a history of the complex proofs and for a proof which has no obvious  $p$ -adic

meromorphic functions given in [GR, pp. 208–209], and then use results of Kiehl and Köpf to deduce the final result.

**Theorem A.1.** *Suppose  $X$  is a proper<sup>21</sup> one dimensional smooth rigid space over a complete local field  $K$  or a compact Riemann surface,  $\mathcal{F} \neq 0$  is a locally free sheaf on  $X$ , and  $D$  is a divisor of positive degree. Then*

$$\lim_{n \rightarrow \infty} \dim_K \mathcal{F}(nD)(X) = \infty,$$

where  $K = \mathbb{C}$  if  $X$  is a Riemann surface.

*Proof.* Let  $E \leq E'$  be divisors on  $X$ , and let  $\mathcal{T} = \mathcal{F}(E')/\mathcal{F}(E)$ . Then

$$0 \rightarrow \mathcal{F}(E)(X) \rightarrow \mathcal{F}(E')(X) \rightarrow \mathcal{T}(X) \rightarrow H^1(X, \mathcal{F}(E)) \rightarrow H^1(X, \mathcal{F}(E')) \rightarrow 0$$

is exact. Moreover, if  $r$  is the rank of  $\mathcal{F}$ , we have

$$\dim_K \mathcal{T}(X) = r \deg(E' - E).$$

Now, for any coherent sheaf  $\mathcal{S}$  on  $X$ , let  $\chi(\mathcal{S}) = \dim_K H^0(X, \mathcal{S}) - \dim_K H^1(X, \mathcal{S})$ . Using the above, we deduce that

$$\chi(\mathcal{F}(D)) - \chi(\mathcal{F}) = r \deg D.$$

The theorem follows. □

**Theorem A.2.** [*p-adic Riemann Existence Theorem*] *Suppose  $X$  is a smooth proper rigid space of dimension one over a complete local field  $K$ . Then  $X$  is isomorphic to the analytification of a complete algebraic curve over  $K$ .*

*Proof.* By the previous theorem, there exists a non-constant map,  $f : X \rightarrow \mathbf{P}_K^1$ , which must be finite since  $X$  is proper of dimension one. By the Direct Image Theorem of Kiehl (see [Ki, Thm. 3.3]), it follows that  $f_*\mathcal{O}_X$  is a coherent sheaf of analytic algebras on  $\mathbf{P}_K^1$ . Subsequently, from [Ko, Satz 4.11 and 5.1] we know that  $f_*\mathcal{O}_X \cong g_*\mathcal{O}_Y$  where  $g$  is a finite morphism from some algebraic curve  $Y$  onto  $\mathbf{P}_K^1$ .

To complete the proof, let  $\mathcal{C}$  be an admissible open covering of  $\mathbf{P}_K^1$  by affinoids. Then  $f^{-1}(\mathcal{C})$  and  $g^{-1}(\mathcal{C})$  are admissible open coverings of  $X$  and  $Y$  by affinoids. Moreover, for each  $U \in \mathcal{C}$  we have

$$A(f^{-1}U) = f_*\mathcal{O}_X(U) \cong g_*\mathcal{O}_Y(U) = A(g^{-1}U).$$

Thus  $f^{-1}U \cong g^{-1}U$  for each  $U \in \mathcal{C}$ , and these isomorphisms are compatible, which implies that  $X \cong Y$ . □

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analogue)

<sup>21</sup>see [BGR, 9.6.2] for definition

## B Supersingular Curves

by Everett Howe

**Theorem B.1.** *For  $p \geq 13$  there is a supersingular elliptic curve  $E$  defined over  $\mathbb{F}_p$  with  $j(E) \neq 0$  or 1728.*

*Proof.* Note that there is always at least one supersingular curve over  $\mathbb{F}_p$ , because the number of curves of trace 0 is given by the Kronecker class number  $H(-4p)$  which is positive (see [Sc] for an account of this). So if  $p$  is a prime for which neither  $j = 0$  nor  $j = 1728$  is supersingular, then there exists a supersingular curve over  $\mathbb{F}_p$  with  $j$  different from 0 and 1728.

If  $p$  is a prime for which  $j = 0$  is supersingular, then  $p$  is inert in the field  $\mathbb{Q}(\sqrt{-3})$ . But then the elliptic curve over  $\mathbb{Q}$  with  $j = 2^4 * 3^3 * 5^3 = 54000$  (which has CM by the order  $\mathbb{Z}[\sqrt{-3}]$ ) reduces to a supersingular curve over  $\mathbb{F}_p$ . (If an elliptic curve over  $\mathbb{F}_p$  is not supersingular then its endomorphism ring tensored with  $\mathbb{Q}$  is an imaginary quadratic field in which  $p$  splits.) Note that 54000 is neither 0 nor 1728 modulo  $p$  for  $p > 11$ .

If  $p$  is a prime for which  $j = 1728$  is supersingular, then  $p$  is inert in the field  $\mathbb{Q}(i)$ . Then the elliptic curve over  $\mathbb{Q}$  with  $j = 2^3 * 3^3 * 11^3 = 287496$  (which has CM by  $\mathbb{Z}[2i]$ ) reduces to a supersingular curve over  $\mathbb{F}_p$ , and 287496 is neither 0 nor 1728 modulo  $p$  when  $p > 11$ .  $\square$

## C Concordance to [CMc]

Due to some shuffling of the material in this paper, some of the references in [CMc] are no longer correct. This problem can be resolved by noting the following changes.

- The reference to §2 on page 265 should be to §2.3.
- Theorem 2.6 is incorrectly referred to as Lemma 3.3 on page 295, and as Lemma 2.3 on page 278.
- Proposition 2.14 is incorrectly referred to as Proposition 3.14 on page 279.
- Proposition 2.34 is incorrectly referred to as Proposition 2.5 on pages 267 and 278.
- Definition 2.35 and Theorem 2.36 are incorrectly referred to as Definition 2.6 and Proposition 2.7 on pp. 279, 292, and 293.
- Proposition 3.6 is incorrectly referred to as Lemma 3.6 on page 278.
- Proposition 4.6 is incorrectly referred to as Corollary 4.6 on page 270.
- Remark 4.7 and Proposition 4.9 are incorrectly referred to as Remark 4.8 and Proposition 4.10 on pg. 272.

- Proposition 7.5 and Remark 7.6 are incorrectly referred to as Proposition 7.4 and Remark 7.5 on pp. 267, 275, 277, and 281.
- Theorem B.1 is cited as “results of E. Howe in §10” on page 262.

## D Index of Important Notation

$K$ , complete non-archimedean valued field	§2
$R_K$ , ring of integers of $K$	
$\mathbb{F}_K$ , residue field of $K$	
$\mathbf{C}$ , completion of an algebraic closure of $K$	
$\mathbf{R}$ , ring of integers of $\mathbf{C}$	
$\overline{\mathbb{F}}$ , residue field of $\mathbf{C}$ and algebraic closure of $\mathbb{F}_K$	
$W(\mathbb{F})$ , Witt vectors of $\mathbb{F}$ for $\mathbb{F} \subseteq \overline{\mathbb{F}}$	
$\mathcal{R}_K$ , value group of $\mathbf{C}^*$	
$\mathbb{C}_p$ , completion of an algebraic closure of $\mathbb{Q}_p$	
$\mathbb{R}_p$ , ring of integers in $\mathbb{C}_p$	
$\Omega_p$ , completion of an algebraic closure of $\overline{\mathbb{F}}_p((T))$	
$\mathbb{N} := \{n \in \mathbb{Z} : n \geq 1\}$ , $\mathbb{N}_0 := \{n \in \mathbb{Z} : n \geq 0\}$	
$B_K(r)$ , $B_K[r]$ , wide open and affinoid disks around 0	
$A_K(r, s)$ , $A_K[r, s]$ , wide open and affinoid annuli	
$C_K[s]$ , the circle $A_K[s, s]$	
$A(X) := \mathcal{O}_X(X)$	
$A^o(X)$ , $A^+(X)$ , subrings of $A(X)$ where $\ f\ _X \leq 1$ and $\ f\ _X < 1$ (when $X$ is a reduced affinoid)	
$\overline{A(X)} := A^o(X)/A^+(X)$	
$\overline{X}$ , canonical reduction of $X$ given by $\text{Spec}(\overline{A(X)})$	
$\text{Red} : X(\mathbf{C}) \rightarrow \overline{X}(\overline{\mathbb{F}})$ , reduction map on $\mathbf{C}$ valued points	
$\text{Red}^{-1}(\tilde{Y})$ , Zariski subaffinoid of $X$ corresponding to affine open $\tilde{Y} \subseteq \overline{X}$	
$\overline{X}^c$ , completion of $\overline{X}$ , non-singular at infinity	
$R(P) := R_X(P)$ , residue class in $X$ of $P \in \overline{X}(\mathbb{F}_K)$	§2.1
$\text{res}_{r,s}$ , canonical residue map on the annulus, $A_K(r, s)$	
$\mathcal{E}(W)$ , $e(W)$ , set of ends, and number of ends, for a rigid space $W$	
$CC(W)$ , set of connected components of a rigid space $W$	§2.2
$H_{DR}^i(W/K)$ , de Rham cohomology of a wide open	§2.3
$g(W)$ , genus of a wide open	
$\mathcal{C}$ , $\mathcal{C}^w$ , semi-stable covering of a wide open or curve	
$U^u$ , underlying affinoid of a wide open $U$ , in a basic wide open pair	
$\Gamma_{\mathcal{C}}$ , graph associated to a semi-stable covering	
$\text{ord}_{A\nu}$ , $\text{ord}_{e\nu}$ , $\text{ord}$ of a function or differential at an annulus or end	§2.4
$\text{Div}(W)$ , divisor group of a wide open	
$\pi_f$ , $\pi_\nu$ , $\pi_{ab}$ , level lowering maps from $X_0(p^n)$ to $X_0(p^m)$	§3
$w_n$ , Atkin-Lehner involution on $X_0(p^n)$	
$K_n(E)$ , canonical subgroup of $E$ of order $p^n$	§3.1
$K(E)$ , (maximal) canonical subgroup of $E$	

$h(E)$ , valuation of Hasse invariant of $E$ (almost)	
$s_n$ , rigid analytic section of $\pi_{0n}$ over $W_n$	
$W_A(p^n)$ , wide open subspace of $X_0(p^n)$ where $\bar{E} \cong A$	
$x_A$ , parameter on $W_A(p)$	
$i(A) :=  \text{Aut}(A) /2$	
$\mathbf{TS}_A, \mathbf{SD}_A$ , too-supersingular and self-dual circles inside $W_A(p)$	
$\mathbf{C}_A, \tau_f$ , special circle of $W_A(p)$ and map to $\mathbf{SD}_A$	
$\mathbf{X}_{a,b}^\pm$ , ordinary affinoids	§3.2
$W_{a,b}^\pm$ , wide open neighborhood of $\mathbf{X}_{a,b}^\pm$	
$Ig(p^n)$ , level $p^n$ Igusa curve	
$(F, A, \alpha)$ , Woods Hole representation of an elliptic curve	§4
$\hat{A}$ , the formal group of $A$	
$B$ , quaternionic order over $\mathbb{Z}_p$ isomorphic to $\text{End}(\hat{A})$	§4.2
$\Phi$ , Gross-Hopkins period map	
$B'$ , special subset of $B^*$	
$w_\rho$ , generalized Atkin-Lehner involution of $\mathbf{SD}_A$ for $\rho \in B'$	
$\mathbf{Y}_A$ , nontrivial affinoid in $W_A(p^2)$	§5
$\mathcal{C}_0(p^2)$ , stable covering of $X_0(p^2)$	
$\mathbf{E}_{1,A}, \mathbf{E}_{2,A}$ , two pullbacks of $\mathbf{Y}_A$ to $X_0(p^3)$	§6
$\mathbf{Z}_A := \pi_{11}^{-1}(\mathbf{SD}_A)$ , affinoid in $W_A(p^3)$ which corresponds to the “bridging component”	
$\tilde{w}_\rho$ , generalized Atkin-Lehner involution of $\mathbf{Z}_A$ for $\rho \in B'$	§7
$\mathcal{D}_\rho^i, \tilde{\mathcal{D}}_\rho^i$ , residue classes of $\mathbf{SD}_A$ and $\mathbf{Z}_A$ invariant under $w_\rho$ and $\tilde{w}_\rho$	
$S_{\sigma,\zeta}, \tilde{S}_{\sigma,\zeta}$ , order $p$ automorphisms of $\tau_f^{-1}(\mathcal{D}_\rho^i)$ and $\tilde{\mathcal{D}}_\rho^i$	§8.2
$V_i(A), U(A)$ , wide open neighborhoods of $\mathbf{E}_{i,A}$ and $\mathbf{Z}_A$	§9
$\mathcal{S}(A)$ , singular residue classes of $\mathbf{Z}_A$	
$\mathbf{X}_S$ , underlying affinoid of $S \in \mathcal{S}(A)$	
$\hat{U}(A)$ , basic wide open refinement of $U(A)$	
$\mathcal{C}_0(p^3)$ , stable covering of $X_0(p^3)$	
$M_K(P)$ , intersection multiplicity at an ordinary double point	§9.1
$w(P)$ , width of the annulus which lifts an ordinary double point	

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