Real-p-Adic Analysis

Lecture 9

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Proposition (Maximum modulus principle). *For A a strict affinoid algebra and $f \in A$, there exists $x \in$ Max A such that $||f||_{spec} = |f(x)|$. Moreover, if $||f||_{spec} = 0$, then f is nilpotent.*

Proof. We may assume A is an integral domainj. By Noether normalization, A is a finite integral extension of some T_d. That means there is an irreducible polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + \ + a_0$$

over T_d such that $P(f) = 0$. From the theory of Newton polygons, for any $x \in$ Max A lying over $y \in$ Max T_d, we have $|f(x)| = \max_i |a_{n-i}(y)|^{1/i}$. We get

$$||f||_{spec} = \max_i ||a_{n-i}||_{spec}^{1/i}.$$

Now suppose $||f||_{spec} = 0$. Then $||a_{n-i}||_{spec} = 0$ for all i.

Corollary. *For A a strict affinoid algebra, the intersection of the maximal ideals of A equals the nilradical of A. In particular, the spectral seminorm is a norm if and only if A is reduced.*

Lemma. *For $\phi : A \to B$ a finite injective homomorphism of affinoid algebras and $f \in A$, one has $||f||_{spec} = ||\phi(f)||_{spec}$.*

Let A^{spec} be the subring of A consisting of those $f \in A$ for which $|f|spec \leq 1$.

"It [is] pure coincidence that the real line coincides with the field of real numbers."

Lemma. *Suppose $\phi : T_d \to A$ is a finite injective K-algebra homomorphism. Then A^{spec} is integral over $\phi(T_d^{spec})$.*

Proof. If A is an integral domain, the proof of the maximum modulus principle yields that any $f \in A^{spec}$ is the root of a polynomial over $\phi(T_d^{spec})$. For the reduction to this case, see [FvdP, Proposition 3.4.5].

Lemma. *For A a strict affinoid algebra under some norm and $f \in A$, one has $||f||_{spec} \leq 1$ if and only if the sequence $\{||f^n||\}$ is bounded.*

Proof. If $\{||f^n||\}$ is bounded

Conversely, suppose $||f||_{spec} \leq 1$. Suppose A is a finite integral extension of T_d. Then $P(f) = 0$ for some monic $P(z) \in T_d^{spec}$.

Corollary. *Let A be a strict affinoid algebra with norm $||\ ||$. Then for $f \in A$,*

$$||f||_{spec} = \lim_n ||f^n||^{1/n} = \rho(f)$$

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