Real-p-Adic Analysis

Lecture 2

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Suppose A is a commutative ring with identity. A function $|\ |\colon A \to \mathbf{R}_+$ such that

$$|0| = 0, \quad |1| = 1$$

$$|f - g| \leq |f| + |g| \text{ and } |fg| \leq |f| \cdot |g|$$

is called a semi-norm. If $|f| = 0$ if and only if $f = 0$, $|\ |$ is called a **norm**. If $|fg| = |f| \cdot |g|$ one says $|\ |$ is **multiplicative**. If $(A, |\ |)$ is complete, one calls it a **Banach algebra**. The seminorm $|\ |$ is **non-Archimedean** if

$$|a + b| \leq \max\{|a|, |b|\}$$

Examples. (i) The trivial norm.

(ii) $\mathbf{Z}$.

(iii) Suppose I is an ideal in a Banach algebra A.

(iv) $\mathbf{Q}_p$.

(iv) If $(k, |\ |)$ is a non-Archimedean complete field with a multiplicative norm and $r_1, \ldots, r_n > 0$ set

$$k(r_1^{-1}T_1, \ldots, r_n^{-1}T_n) = \{\sum_{\nu \in \mathbf{N}^n} a_\nu T^\nu\colon |a_\nu| r^\nu \to 0 \text{ as } |\nu| \to \infty\}$$

$$|| \sum_{\substack{\nu \in \mathbf{N}^n \\ \nu=0}}^{\infty} a_\nu T^\nu || = \max_\nu |a_\nu| r^\nu.$$

We will call this more simply $k(r^{-1}T)$.

Suppose $(A, || \, ||)$ is a Banach ring the **spectrum** of A, $\mathcal{M}(A)$, is the set of all multiplicative semi-norms $| \, |$ on A such that $\exists C > 0$

$$|f| \leq C||f||,$$

with the weakest topology so that the functions $| \, | \to |f|$ are continuous. I.e., the sets

$$\{| \, | \in \mathcal{M}(A) \colon -\epsilon < |f| - x < \epsilon\}$$

are open for all $f \in A$, $x \in \mathbf{R}$ $\epsilon > 0 \in \mathbf{R}$.

If A is a quotient of $k(r^{-1}T)$. Then $(\mathcal{M}(A), A)$ is called a ***k*-affinoid**. If $r = (1, \ldots, 1)$, A is called **strictly *k*-affinoid**. A pair $(V, \alpha : A \to B)$ is called an **affinoid domain** in $X := \mathcal{M}(A)$, if $V \subseteq \mathcal{M}(A)$, α is a bounded homomorphism which induces a homeomorphism $\mathcal{M}(B) \to V$ such that if $\beta \colon A \to C$ is a bounded homomorphism such that $\beta^* \mathcal{M}(C) \subseteq V$, there exists a unique bounded homomorphism $: B \to C$ such that

$$\beta = \gamma \circ \alpha.$$

Example. If $r = 1$ then $A := \mathbf{C}_p(r^{-1}T)$ is also called $\mathbf{C}_p\langle T\rangle$. Then if $c \in \mathbf{C}_p$, $|c| \leq 1$ the map

$$f \to |f(c)|$$

is an element $| \, |_c$ of $X := \mathcal{M}(A)$. Now suppose $s \leq 1$. Then

$$f \to \sup_{|x-c| \leq s} |f(x)|$$

is also an element $| \, |_{c,s}$ of X. Let $B = \mathbf{C}_p(s^{-1}T)$. Then B is strictly $\mathbf{C}_p$-affinoid if and only if $s \in p^{\mathbf{Q}}$. Also,

$$f(T) \to f(T - c)$$

is a homomorphism and $f^* \mathcal{M}(B)$ onto a subdomain $B[c, s]$ of X.

Exercises (Due next Friday)

1. Verify some of the formal stuff that I didnt check, like the fact that multiplication on $(K^{alg})^*/q^{\mathbf{Z}}$ translates into addition on the Tate curve.

2. Check all the statements I made about the endomorphism ψ_1 of an elliptic curve, e.g., that it satisfies a quadratic polynomial whose roots are nonreal.