Real-p-Adic Analysis

Lecture 19

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Curves

Suppose $f(x,y) \in K[x,y]$ and f is not constant. Let $C :=$ Spec $K[x,y]/(f)$ and $X :=$ Max A where $A = K\langle x,y\rangle/(f)$. Suppose $f(x,y) \in R[x,y]$. Then C reduces to $\overline{C} :=$ Spec $\mathbf{F}[x,y]/(\bar{f})$.

Lemma. *if A and $\mathbf{F}[x,y]/(\bar{f})$ are reduced, $A^o = B := R[x,y]/(f)$.*

For $z \in \overline{C}(\mathbf{F})$, let $I_z = \{a \in A^o \colon \bar{a} \in m_z\}$ and $R(z) = \{x \in X \colon |a(x)| < 1, \forall a \in I_z\}$.

Lemma. *Suppose $z \in \overline{C}(\mathbf{F})$ is a smooth point. Then $R(z)$ is isomorphic to $B_K(0,1)$.*

Proof. Suppose $z = (0,0)$ and $f_2(0,0) \in R^*$. Then $f_2(x,0) \in R[[x]]^*$ and $f(x,0) \in xR[[x]]$. Suppose $a \in R[[x]]$, then

$$f(x,a+h) = f(x,a) + f_2(x,a)h + \cdots + b_n(x)h^n + \cdots$$

where $b_n(x) \in R[[x]]$. Suppose

$$f(x,a_n(x)) \in x^n R[[x]] \text{ and } f_2(x,a_n(x)) \in R[[x]]^*.$$

"It [is] pure coincidence that the real line coincides with the field of real numbers."

Proper Curves

Lemma. *Suppose A is an affinoid algebra over K and $A \to B$ is finite. Then B is an affinoid algebra over K.*

Proof.

Now suppose Z is complete curve over K and $f_1, \ldots, f_n$ are non-constant functions on Z such that affinoids $X_i = \{x \in Z : |f_i(x)| \leq 1\}$ cover Z.

Suppose Z has a model with good reduction $\bar{Z}$. Then if $z \in \bar{Z}$, $R(z)$ is a disk and $X := Z \backslash R(z)$ is an affinoid.

In fact, $\mathcal{O}_X(X)$ is isomorphic to the completion of $\mathcal{O}_Z(Z\backslash\{z\})$ w.r.t. $|f|_X = \sup\{|f(x)| : x \in X\}$. Suppose $h_1 : R(z) \to B_K(0,1)$ is an isomorphism. Then $X_1 := Z \backslash h_1^{-1}(B_K[0,r])$ is a strongly admissable open in Z.

and

$$X_{1\,2} = R(z) \backslash h_1^{-1}(B_K[0,r])$$

is an annulus ($\cong A_K(r,1) \cong A_K(1,1/r)$). Suppose $a \in K$, $|a| = r$ and Z' is another complete curve with good reduction and $z' \in \bar{Z}'$. Suppose $h_2 : R(z') \cong B_K(0,1)$, $X_2 = Z' \backslash h_2^{-1}(B_K[0,r])$ and $X_{2,1} = R(z') \backslash h_2^{-1}(B_K[0,r])$. Let

$$\phi_{1,2}(x) = h_2^{-1}(a/h_1(x)),$$

for $x \in X_{1,2}$.