

Galois Theory

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More Final Problems

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1. Let p be a prime and $r \geq 1$ an integer. Show $x^{p^r} - x$ is the product of all the monic irreducible polynomials over \mathbf{F}_p of degree dividing r .
2. Let $F_n(x) = \sum_{i=0}^n x^i/i!$. Show the Galois group of the splitting field of $F_n(x)$ over \mathbf{Q} is isomorphic to a subgroup of A_n if $4|n$.
3. True-False. If false, give a counterexample.
 - (i) Every field has a normal extension of degree 2.
 - (ii) The intersection of a quadratic and a cubic extension of a field K is K .
 - (iii) A normal extension with a solvable Galois group is a radical extension.
 - (iv) A non-normal extension of a non-normal extension is not normal.
 - (v) Two extensions of \mathbf{Q} in \mathbf{C} are isomorphic if and only if they are equal.
4. Suppose F and K are normal extensions of \mathbf{Q} in \mathbf{C} . Show $\text{Gal}(\mathbf{Q}(F \cup K)/\mathbf{Q})$ injects into $\text{Gal}(F/\mathbf{Q}) \times \text{Gal}(K/\mathbf{Q})$.
5. Find an extension of \mathbf{Q} with Galois group $C_3 \times C_3 \times C_3$. (This group might be changed if this question appears on the exam.)smk
6. Suppose $a, b \in \mathbf{Q}^*$. show $\mathbf{Q}(\sqrt{a}) \cong \mathbf{Q}(\sqrt{b})$ if and only if $ab \in \mathbf{Q}^2$.
7. Suppose k is an infinite field of characteristic p . Let $K = k(x, y)$ and $F = K^p$. Show K/F is a finite extension of fields with infinitely many subextensions.
8. Show the Galois group of $f(x) = x^3 + pqa x + pq^2 b$ is S_3 where p and q are distinct primes, a and b are integers such that $(qb, 2ap) = 1$. (Hint: Show f is irreducible and compute its discriminant.)
9. True/False If false, give a counter-example.
 - (i) The Galois group of a finite normal separable extension of fields of characteristic

p has order prime to p .

(ii) Every finite normal extension of a field F is a splitting field of a polynomial in $F[x]$.

(iii) If K is an extension of F and $K = F(\alpha_1, \dots, \alpha_n)$, the transcendence degree of K/F is the maximum number of algebraically independent over F elements in $\{\alpha_1, \dots, \alpha_n\}$.

(iv) If K/F is a normal extension of degree at least 5, K/F is not solvable.

(v) The intersection of normal extensions is a normal extension.

10. Suppose K is an extension of a field F and A and B are normal extensions of F in K . Show the smallest field in K containing A and B is a normal extension of F .

11. Suppose K/F is normal and separable, $G =: \text{Gal}(K/F)$ is a finite simple group which contains a non-trivial subgroup H of index k . Show K is the splitting field over F of a polynomial of degree k . (Hint: Use the minimal polynomial of a primitive element of K^H .)

12. Let $g(E) = E^6 + 2E^3 - 1$ and K be a splitting field of g over \mathbf{Q} . Determine $G = \text{Gal}(K/\mathbf{Q})$. Find all subfields of K .

13. True/False. If false give a counterexample. Suppose K/F is finite normal and separable. Let $G = \text{Gal}(K/F)$.

- (i) The number of subextensions of F in K is finite.
- (ii) If G is Abelian, every subextension of F in K is normal over F .
- (iii) If G is solvable, there exists a positive integer d and $\{\alpha_1, \dots, \alpha_n\} \subseteq K$ such that $\alpha_i^d \in F$ and $K = F(\alpha_1, \dots, \alpha_n)$.
- (iv) If K is finite, the number of subextensions of F in K is at most $\log_{|F|} |K|$.
- (v) if $f(x)$ is an irreducible polynomial over F with three distinct roots in K , α, β, γ , $F(\alpha, \beta)$ is isomorphic to $F(\beta, \gamma)$.

14. Determine all irreducible cubics over \mathbf{F}_3 .

15. Suppose $f(x)$ is a degree 4 polynomial over \mathbf{Q} and K is a splitting field of f . Suppose $f(x) = \prod_{i=1}^4 (x - \alpha_i)$ over K . Show

$$[\mathbf{Q}((\alpha_1 - \alpha_2)(\alpha_3 - \alpha_4)) : \mathbf{Q}(\alpha_1\alpha_2 + \alpha_3\alpha_4)] = 2$$

if $[K : \mathbf{Q}] = 24$. (Hint: Show $\text{Gal}(K/\mathbf{Q}((\alpha_1 - \alpha_2)(\alpha_3 - \alpha_4)))$ is a normal subgroup of order 4 and $\text{Gal}(K/\mathbf{Q}(\alpha_1\alpha_2 + \alpha_3\alpha_4))$ is a Sylow 2-subgroup.)

16. True/False

- (i) If F/K and L/K are finite normal separable extensions such that $\text{Gal}(F/K)$ is not isomorphic to a quotient of $\text{Gal}(L/K)$, then F/K is not a subextension of L/K .
- (ii) Any two finite fields are isomorphic to subfields of a third finite field.
- (iii) The additive group of a finite field is cyclic.
- (iv) Every finite Abelian group is isomorphic to the Galois group of infinitely many finite normal extensions of \mathbf{Q} .
- (v) The extension $\mathbf{Q}(\sqrt{-1}, \sqrt{2})/\mathbf{Q}$ has three subextensions.