Galois Theory

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Lecture 5

Algebraic Extensions

Theorem. *The set L of elements in E algebraic over F is a field extension of F.*

Proof. $F \subseteq L$.

Suppose $\alpha, \beta \in L$. First $F(\alpha, \beta) = (F(\alpha))(\beta)$. Also β is algebraic over $F(\alpha)$. We know

$$[F(\alpha, \beta) : F] = [F(\alpha, \beta) : F(\alpha)] \cdot [F(\alpha) : F].$$

Suppose $\alpha \in E$ is algebraic over F. One calls $[F(\alpha) : F]$ the degree of α over F. Let $I = \{f \in f[x]: f(\alpha) = 0\}$. The monic polynomial $g_\alpha(x)$ such that $I = (g_\alpha(x))$ is called the **minimal polynomial** of α.

Lemma. $g_\alpha(x)$ *is irreducible and* $\deg g_\alpha(x) = \deg(\alpha)$.

Proof.

Homework for Monday

A. Show there are no extensions of $\mathbf{R}$ of finite odd degree.

B. Suppose E is an extension field of F and $e \in E$. Show $F(e) \cong F(x)$, the rational functons of F, if and only if e is transcendental over F.