Galois Theory

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Lecture 3

Correct the typo in 2.2.7.

$$A^3 + 3BA = 2Z$$

Suppose $B = E(A+E)$, i.e. $A = B/E - E$. Then, $E^6 + 2ZE^3 = B^3$ so $E = \sqrt[3]{-Z \pm \sqrt{Z^2 + B^3}}$.

Show every polynomial over $\mathbf{R}$ of odd degree has a real root.

Find an example where Viète's method gives a non-real root. Find it anyway.

I am assuming you know what rings, fields, ideals, homomorphisms and kernels are.

Field Extensions

Suppose E is a field and F is a subfield then one says E is an **extension field** of F.

Examples.

Lemma. *If E is an extension of F, E is a vector space over F.*

Its dimension is called the degree of E over F which we denote $[E : F]$.

Proposition. *If F is a field and $g(x) \in F[x]$ is irreducible of positive degree d then $E =: F[x]/(g(x))$ is a extension field of the image of F (which is isomorphic to F) of degree d.*

Proof. Suppose $\pi\colon F[x] \to E$ is the natural map. Suppose $s \in E$, $s \neq 0$.

More *Examples.* . I will call $\mathbf{Z}/p\mathbf{Z}$ $\mathbf{F}_p$.

$\mathbf{F}_2[x]/(x^2 - x + 1)$.

$\mathbf{F}_p[x]/(x^p - x - 1)$

Theorem. *If $F \subseteq L \subseteq E$, then $[E : F] = [E : L] \cdot [L : F]$.*

Proof.

$\mathbf{Q}$ is the only proper subfield of $\mathbf{Q}[x]/(x^3 - 2)$.

An element $\alpha \in E$ such that $f(\alpha) = 0$ for some non-zero $f \in F[x]$ is said to be **algebraic over** F. If $A \subseteq E$, $F(A)$ is smallest extension of F in E containing A.

Proposition. *An element $\alpha \in E$ is algebraic over F iff $[F(\alpha) : F] < \infty$*

Proof.

Homework for next time

Read §4.1-§4.3. Show every element of $\mathbf{C}$ has a square root in $\mathbf{C}$. Do 4.4. (Study means determine.)