Galois Theory

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Lecture 32

Trascendence

The elements $\alpha_1, \ldots, \alpha_n$ in an extension F of K are said to be **algebraically independent** over K if they satisfy no polynomial equation. For example, if $x_1, \ldots, x_n$ be variables and $N_n = \mathbf{Q}(x_1, \ldots, x_n)$, then $\alpha_1, \ldots, \alpha_n$ are algebraically independent over $\mathbf{Q}$.

The set $\alpha_1, \ldots, \alpha_n$ is said to be a **transcendence basis** for F/K if F is algebraic over $K(\alpha_1, \ldots, \alpha_n)$ and n called the **transcendence degree** of F/K. Thus the transcendence degree of $N_n/\mathbf{Q}$ is n.

Proposition. *Transcendence degree is well defined.*

Proof.

Last time we set

$$\prod_{i=1}^{n}(T - x_i) = \sum_{j=1}^{n} e_j(x_1, \ldots, x_n)T^j$$

and let $E_n = \mathbf{Q}(\{e_j(x_1, \ldots, x_n) : 1 \leq j \leq n\}) \subset N_n$. We proved,

Proposition. *N_n/E_n is normal and $Gal(N_n/E_n) \cong S_n$.*

Corollary. $N_n \cong E_n$.