

**Galois Theory**  
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**Lecture 28**

**The Symmetric groups**

Let

$$P(X) = X^5 - 10X + 5.$$

Last time we saw the Galois group of the splitting field of this polynomial contains a 5-cycle and a transposition. This is enough as,

**Theorem.** *Suppose  $p$  is a prime. Then  $S_p$  is generated by any  $p$ -cycle and any transposition.*

**Lemma.** *Suppose  $\sigma$  and  $\tau = (a_1 \dots a_k)$  are in  $S_n$ . Then*

$$\sigma\tau\sigma^{-1} = (\sigma(a_1) \dots \sigma(a_k)).$$

*Proof.*

**Lemma.**  *$S_n$  is generated by*

$$(12) \text{ and } (12 \dots n).$$

*Proof.*

**Corollary.** *If  $p$  is prime,  $G$  is a subgroup of  $S_p$ ,  $p \mid |G|$  and  $G$  contains a transposition then  $G = S_p$ .*

*Proof.*

## Other Examples

Let

$$F_n(X) = \sum_{i=0}^n \frac{X^i}{i!}$$

**Theorem.** (Schur) The Galois Group of the splitting field of  $F_n(X)$  is isomorphic to  $S_n$  if  $4 \nmid n$  and  $A_n$  otherwise.

Suppose  $P(X) \in K[X]$ ,  $P(X) = a \prod_{i=1}^n (X - \alpha_i)$ . Let

$$D_P = (-1)^{n(n-1)/2} \frac{\prod_i P'(\alpha_i)}{a^n}.$$

**Proposition.** The Galois group of  $P$  is contained in  $A_n$  if and only if  $D_P \in K^2$ .

*Proof.*

$$D_P = \prod_{j>i} (\alpha_j - \alpha_i)^2.$$

Suppose  $a < b$ .

$$\prod_{j>i} (\alpha_j - \alpha_i) = (\alpha_b - \alpha_a) \prod_{a<i<b} (\alpha_b - \alpha_i)(\alpha_i - \alpha_a) E_{ab}$$

*Example.*  $F_5(X)$

**Read** Chapter 13

### Homework for Monday

Prove the Galois group of the splitting field of  $F_n(X)$  is contained in  $A_n$  if and only if  $4|n$  and has order divisible by  $n$  if  $n$  is prime.