

Galois Theory

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Lecture 26

Radical Extensions

An extension K/F is **radical** if there exists a sequence of subextensions

$$F = K_0 \subset K_1 \subset \cdots \subset K_r = K,$$

and an integer d such that $K_{i+1} = K_i(\{\alpha_{i1}, \dots, \alpha_{ik_i}\})$ for some $\alpha_{ij} \in K_{i+1}$ where $\beta_{ij} := \alpha_{ij}^d \in K_i$.

Proposition. *If K/F is radical K is contained in a Galois radical extension of F .*

Proof. Suppose $r = 2$. Let L be

$$K_1(\mu_d, \{(\sigma(\beta_{1j}))^{1/d} : \sigma \in \text{Gal}(K_1(\mu_d)/F)\}).$$

Theorem. *If K/F is Galois, K is contained in radical extension of F if and only if $G := \text{Gal}(K/F)$ is solvable.*

Proof. Suppose $H_r = (e) \subset H_{r-1} \subset \cdots \subset H_0 = G$ is a sequence of subgroups such that H_i is normal in H_{i-1} and H_{i-1}/H_i is Abelian of exponent dividing d . Now let $L_0 = K$ and

$$L_{i+1} = K^{H_i}(\mu_d) \text{ for } i \geq 0.$$

Suppose $K \subseteq L$ and L/F is Galois and radical. Let $E = L(\mu_d)$ where d

A Non-Solvable Example

$$P(X) = X^5 - 10X + 5$$

$$P(2^{1/4}) = -4.5\dots \text{ and } P(-2^{1/4}) = 14.5\dots$$

Theorem. *Suppose $f(x)$ is an irreducible polynomial over \mathbf{Q} of prime degree p with exactly two non real roots. Then the Galois group of its splitting field of f is isomorphic to S_p .*

Proof.

Read §12.1-§12.2.

Homework for Friday

Think about problem 11.1.

Homework for Monday

Do 12.1 1). Give the proof of the above proposition for arbitrary r .