Galois Theory

Robert F. Coleman

Lecture 25

Solvable Groups

A group G is said to be **solvable** if and only if there a sequence of subgroups $H_0 = (e) \subset H_1 \subset \cdots \subset H_r = G$ such that H_i is normal in H_{i+1} and H_{i+1}/H_i is Abelian.

Example. S_4

Lemma. *If G is solvable and H is either a sub or quotient group of G, then H is solvable.*

Let G be a group. The intersection of all subgroups of G with Abelian quotient $D(G)$ is called the **commutator subgroup** of G. If $\sigma, \tau \in G$, the **commutator** of σ and τ is

$$[\sigma, \tau] =: \sigma\tau\sigma^{-1}\tau^{-1}.$$

Lemma. *if G is solvable $D(G) \neq G$.*

Proof.

Lemma. *$D(G)$ is generated by the commutators of G.*

Proof.

The Alternating Groups

Suppose $n \geq 3$.

$$(1, 2, \ldots, n)(n, n-1, 1) = (1, 2, \ldots, n-2)$$

Theorem. *A_n is generated by 3-cycles.*

Proof.

Theorem. *S_n is not solvable for $n \geq 5$.*

Proof.

Homework for Monday

Suppose $f(x) \in \mathbf{Q}[x]$ has degree 4 and K is he splitting field in $\mathbf{C}$ of f. Suppose $f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)$ and $G =: \text{Gal}(K/\mathbf{Q}) \cong S_4$. Let N be the normal subgroup of G of order 4. Find generators for K^N.