Galois Theory

Robert F. Coleman

Lecture 17

Our Example

Let $f(x) = x^3 + 3Bx - 2Z$, $B, Z \in \mathbf{Z}$ and $g(E) = E^6 + 2ZE^3 - B^3$. Suppose f is irreducible, $B > 0$ and $B^3 + Z^2$ is not a square. If α is a root of g, $B/\alpha - \alpha$ is a root of f. The 6 roots of g are $j^i\alpha$ and $-j^iB/\alpha$, $0 \le i \le 2$. Note, $(\alpha^3 + Z)^2 = B^3 + Z^2$ and $(j - j^{-1})^2 = -3$. Let K be the splitting field of g and $L \subset K$ the splitting field of f. We know $[K : \mathbf{Q}] = 12$ and $[\mathbf{Q}(\alpha) : \mathbf{Q}] = [L : \mathbf{Q}] = 6$. We also know $\mathrm{Gal}(K/\mathbf{Q})$ is generated by σ, τ and γ where $\sigma\alpha = j\alpha$, $\sigma j = j$; $\tau\alpha = -B/\alpha$; $\gamma\alpha = \alpha$, $\gamma j = 1/j$.

Claim: Let θ be a root of f. Then the splitting field L of f is

$$\mathbf{Q}(\theta, \sqrt{-3(B^3 + Z^2)})$$

and $\mathrm{Gal}(L/\mathbf{Q}) \cong S_3$ iff $-3(B^3 + Z^2)$ is not a square. (By 7.1.1, this is true even when $B \leq 0$.)

Homework for Monday

Figure out what happens in the above example when $B^3 + Z^2$ is a square.