

Galois Theory

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Lecture 15

Mid-Term

1. True-False. If false, give a counterexample. Let K be a field.

(i) A normal extension of a normal extension of K is a normal extension of K .

(ii) The extension $\mathbf{Q}(\sqrt[2]{-3}, \sqrt[3]{2}, \sqrt[3]{3})/\mathbf{Q}$ is normal.

(iii) Every subextension of a normal extension of K is a normal extension of K .

(iv) All fields are normal extensions of themselves.

(v) If $\alpha \neq 0$ is algebraic of degree d and $f(x) \in \mathbf{Q}[x]$ is the minimal polynomial for α then $x^d f(x^{-1})/f(0)$ is the minimal polynomial for α^{-1} .

(vi) All quadratic extensions of \mathbf{Q} are isomorphic.

(vii) All irreducible cubics over \mathbf{Q} have a non real root.

If $f(x) = x^3 - 3ax^2 + b$, f has local extrema at 0 and $2a$. $f(2a) = -4a^3 + b < -0$ if $4a^3 > b$. So if $a = b = 2$, f is irreducible and has 3 real roots.

(viii) If $\alpha \in \mathbf{C}$ and $\mathbf{Q}(\alpha) = \mathbf{Q}[\alpha]$, then α is algebraic.

(ix) 4 divides the degree of the splitting field of quartic.

(x) The equation $x^n + y^n = z^n$ has a non-trivial solution over some algebraic extension of \mathbf{Q} .

2. Let $F_n = \mathbf{Q}(\sqrt[n]{2})$. Show if p and q are distinct primes $F_p \cap F_q = \mathbf{Q}$.

$$[F_n : \mathbf{Q}] = n$$

3. Show the intersection of two splitting fields over \mathbf{Q} in \mathbf{C} is splitting field over \mathbf{Q} .

Splitting fields = Normal finite extensions.

4. Determine whether or not the following polynomials are irreducible over \mathbf{Q} :

(i) $x^3 + x + 1$

(ii) $x^3 + 6x^2 - 25$

(iii) $x^4 + 2x^3 + 3x^2 + 2x + 1$

(iv) $\sum_{n=0}^5 \frac{x^n}{n!}$.

5. Let $g(E) = E^6 + 2E^3 - 1$ and K be a splitting field of K over \mathbf{Q} . List as many distinct subfields of K as you can. Prove they are distinct.

Let L be the splitting field of $x^3 + 3x - 2$ in K and $\alpha = \sqrt[3]{-1 + \sqrt{2}}$. Then we have $\mathbf{Q}, K, L, \mathbf{Q}(\alpha), \mathbf{Q}(j\alpha), \mathbf{Q}(j^2\alpha), \mathbf{Q}(\alpha-1/\alpha), \mathbf{Q}(j^2\alpha-1/j^2\alpha), \mathbf{Q}(j\alpha-1/j\alpha), \mathbf{Q}(\sqrt{-3}), \mathbf{Q}(\sqrt{2}), \mathbf{Q}(\sqrt{-6}), \mathbf{Q}(\sqrt{-3}, \sqrt{2}), \dots$

Groups

If G is a finite group and $p^k \mid |G|$ where p is a prime then G has a subgroup S of order p^k .

Take home quiz. Write up a solution and hand it on Friday.

Proof. Suppose $E = N^H$ is normal. Then we have,

$$0 \rightarrow H \rightarrow G \rightarrow \text{Gal}(E/K) \rightarrow 0.$$

For Monday. Do 8.6 and 8.8.