

Galois Theory

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Lecture 13

Galois Groups

Suppose L/K is an extension of fields. Then the set of K -homomorphisms from L onto L is a group denoted $\text{Gal}(L/K)$.

If H is a subgroup of $\text{Gal}(L/K)$ and E is a subextension of L let

$$L^H = \{\ell \in L: \ell^h = \ell \text{ for } h \in H\}$$

and

$$G_E = \{g \in G, \ell^g = \ell \text{ for } \ell \in E\}.$$

We know, L^H is a subextension of L and G_E is a subgroup of G . Moreover, $G_{L^H} \supseteq H$ and $L^{G_E} \supseteq E$.

Theorem. If N/K is normal and separable of finite degree, $G_{N^H} = H$ and $N^{G_E} = E$. I.e., the maps $\phi: H \mapsto N^H$ and $\sigma: L \mapsto G_L$ between subgroups of G and subextensions of N/K are inverses.

Proposition. If $|H| < \infty$, L/L^H is normal and separable, and $[L : L^H] = |H|$

Proof.

Proposition. If $[L : K] < \infty$, $|\text{Gal}(L/K)| = [L : K]$ with equality if and only if L/K is normal and separable.

Proof.

Proof of Theorem.

Example. Recall if $f(x) = x^3 + 3Bx - 2Z$, Viète tells us to look at $g(E) = E^6 + 2ZE^3 - B^3$. ■

Suppose $B > 0$. If α is a root of g , $\alpha \neq 0$ and $B/\alpha - \alpha$ is a root of f . The 6 roots of g are $j^i \alpha$ and $-j^i B/\alpha$, $0 \leq i \leq 2$. Note, $(\alpha^3 + Z)^2 = B^3 + Z^2$ and $(j - j^{-1})^2 = -3$.

The Fundamental Theorem of Algebra

Theorem. \mathbf{C} is algebraically closed.

Sylow's Theorem. If G is a finite group and $p^k \mid |G|$ where p is a prime then G has a subgroup S of order p^k .

If $(|S|, |G|/|S|) = 1$, then S is called a p -**Sylow** subgroup.

Suppose K is a finite algebraic extension of \mathbf{R} . Let N be a normal closure of K/\mathbf{R} . Let $G = \text{Gal}(N/\mathbf{R})$ and S the 2-Sylow subgroup of G . Let $F = K^S$. Then

$$[F : \mathbf{R}] = [N : \mathbf{R}]/[K : F] = |G|/|S|.$$

Which is odd so $F = \mathbf{R}$ and $G = S$.

Suppose $S = 2^k$. Claim $k \leq 1$.

Homework for Monday

Read §8.3-§8.6.

Deduce the above version Sylow's Theorem from Problems 8.1 and 8.2. Verify that the 6 roots in the above example are distinct. Do 8.4 and 8.5.