

Algebraic Number Theory

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Lecture 9

Unramified Extension of Local Fields

Suppose L/K is finite extension of local fields with valuations v_L and v_K and residue fields k_L and k_K such that $v_L|_K = e_{L/K}v_K$. We say L is **unramified** over K if $e_{L/K} = 1$. Then

Proposition. $[L : K] = e_{L/K}[k_L : k_K]$.

Lemma. If L/K is unramified, L/K is Galois and $\text{Gal}(L/K) \cong \text{Gal}(k_L/k_K)$.

We proved this assuming $\text{Gal}(L/K)$ acts continuously...

Corollary. If L/K is unramified, $\exists! \sigma \in \text{Gal}(L/K)$, $\sigma(a) \equiv a^{q_K} \pmod{m_L}$ for $a \in m_L$.

Back to Lubin-Tate

Let $\pi \in A$ be a uniformizing parameter. Recall \mathcal{F}_π is the set of $g(T) \in A[[T]]$ such that (a) $g(T) \equiv \pi T \pmod{T^2}$ and (b) $g(T) \equiv T^q \pmod{\pi}$. We know

Proposition. Suppose $f \in \mathcal{F}_\pi$. Then there exists a unique formal group F_f over A such that $f(F_f(X, Y)) = F_f(f(X), f(Y))$.

Also, there is an injective homomorphism $a \mapsto [a]_f$ from A into $\text{End}F_f$ and if $f, g \in \mathcal{F}_\pi$, there is an isomorphism $\iota_{f,g}: F_f \rightarrow F_g$ such that $\iota_{f,g} \circ [a]_f = [a]_g \circ \iota_{f,g}$.

The extension K_π .

Fix a separable closure K_s of K .

Lemma. Suppose $f, g \in \mathcal{F}_\pi$. Then

$$K(\{a \in m_{K_s}: [\pi^n]_f(a) = 0\}) = K(\{a \in m_{K_s}: [\pi^n]_g(a) = 0\})$$

Proof.

Let K_π^n be this field and $K_\pi = \bigcup_n K_\pi^n$.

Proposition. $\text{Gal}(K_\pi/K) \cong A^*$.

Proof. Pick $f \in \mathcal{F}_\pi$. Let E_f be the A -module

$$\bigcup_n \{a \in K^s : [\pi^n]_f(a) = 0\}$$

Claim $E_f \cong K/A$.

We can choose $f(T) = \pi T + T^q$.

Proof of proposition.

Corollary. π is a norm from K_π^n so K_π^n/K is totally ramified.

Proof.

The Extension(s) $K_{nr}K_\pi$

Let K_{nr} be the maximal unramified extension of K and $L_\pi = K_\pi K_{nr}$. We want to show L_π is the maximal Abelian extension of K .

Let \hat{K}_{nr} be the completion of K_{nr} . Extend the Frobenius auto. σ to \hat{K}_{nr} . Suppose $\pi' = \pi u$ is a uniformizing parameter.

Lemma. Suppose $f \in \mathcal{F}_\pi$. There exists $\phi \in \hat{A}_{nr}[[X]]$ such that $\phi(X) \equiv \epsilon X \pmod{\deg 2}$, $\epsilon \in \hat{A}_{nr}^*$ and $\phi^\sigma = \phi \circ [u]_f$.

Corollary. If $g = \phi \circ f \circ \phi^{-1}$, $g \in \mathcal{F}_{\pi'}$.