

Algebraic Number Theory

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Lecture 6

Extensions of local fields

Theorem. Suppose K is a local field with valuation v and L is a finite extension of K . Then L is a local field with valuation extending v .

Proof. Let $n = [L : K]$. For x in L set

$$w(x) = v(N_{L/K}(x))/n.$$

Let E be the maximal separable subextension of L/K , $f = [E : K]$ and $e = n/f$. Assume L is quasi-Galois, then

$$N_{L/K}(x) = \left(\prod_{\sigma \in \text{Aut}(L/K)} \sigma(x) \right)^e.$$

Unramified Extension of Local Fields

Suppose L/K is finite extension of local fields with valuations v_L and v_K and residue fields k_L and k_K such that $v_L|_K = e_{L/K}v_K$. Then

Proposition. $[L : K] = e_{L/K}[k_L : k_K]$.

Proof.

We say L is **unramified** over K if $e_{L/K} = 1$.

Lemma. If L/K is unramified, L/K is Galois and $\text{Gal}(L/K) \cong \text{Gal}(k_L/k_K)$.

We'll need

Hensel's Lemma. Suppose $g(x) \in A_K[x]$, $a \in A_K$, $g(a) \equiv 0 \pmod{m_K}$ and $g'(a) \not\equiv 0 \pmod{m_K}$. Then $\exists \alpha \in A_K$, $g(\alpha) = 0$

Proof.

Proof of Lemma.

Corollary. If L/K is unramified, $\exists! \phi \in \text{Gal}(L/K)$, $\phi(a) \equiv a^{q_K} \pmod{m_L}$ for $a \in m_L$.

Back to Lubin-Tate

Let $\pi \in A$ be a uniformizing parameter. Let \mathcal{F}_π be the set of $g(T) \in A[[T]]$ such that (a) $g(T) \equiv \pi T \pmod{T^2}$ and (b) $g(T) \equiv T^q \pmod{\pi}$.

Proposition. Suppose $f \in \mathcal{F}_\pi$. Then there exists a unique formal group F_f over A such that $f(F_f(X, Y)) = F_f(f(X), f(Y))$.

There is an injective homomorphism $a \mapsto [a]_f$ from A into F_f and if $f, g \in \mathcal{F}_\pi$, there is an isomorphism $\iota_{f,g}: F_f \rightarrow F_g$ such that $\iota_{f,g} \circ [a]_f = [a]_g \circ \iota_{f,g}$.

The extension K_π .

Fix a separable closure K^s of K .

Lemma. Suppose $f, g \in \mathcal{F}_\pi$. Then

$$K(\{a \in K^s: [\pi^n]_f(a) = 0\}) = K(\{a \in K^s: [\pi^n]_g(a) = 0\})$$

Proof.

Let K_π^n be this field and $K_\pi = \bigcup_n K_\pi^n$.