Algebraic Number Theory

Robert F. Coleman

Lecture 27

Rationality

We know $Z(X,T) \in \mathbf{Z}[[T]]$, the coefficient of T^i in $Z(X,T)$ is at most q^{ni} and $Z(X,T)$ is meromorphic. Now we want t prove,

Theorem. *$Z(X,T)$ is rational.*

We will use

Lemma. *Suppose $F(T) = \sum_{i=0}^{\infty} a_i T^i \in L[[T]]$. Let $A_{s\,m}$ be the $m+1 \times m+1$ matrix $(b_{i\,j})_{0\le i,j\le m}$, where $b_{i\,j} = a_{s+i+j}$. Then $F(T)$ is the expansion at 0 of a rational function if and only if there exists $m \ge 0$ such that $\det A_{s\,m} = 0$ for s sufficiently large.*

and

Basic Fact. *If $a \in \mathbf{Q}^*$*

$$\prod_p |a|_p = 1.$$

Take $F(T) = Z(X,T)$. Then

$$|\det A_{s\,m}|_\infty \le (m+1)! q^{n(m+1)s} q^{nm(m+1)}$$

Now by Weiersträss Preperation, for any $R > 0$, $Z(X,T) = F_R(T)/P_R(T)$, where

$$P_R(T) = 1 + c_1 T + \cdots + c_e T^e$$

$$F_R(T) = \sum_{i=0}^{\infty} b_i T^i$$

where $|b_i|_p R^i \to 0$. In particular,

$$b_{j+e} = a_{j+e} + c_1 a_{j+e-1} + \cdots + c_e a_j$$

We get, for $m \geq 2e$ and large s,

$$|\det A_{s\,m}|_p \leq R^{-s(m+1-e)}$$

Take $R = q^{2n}$.