Algebraic Number Theory

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Lecture 26

Lemma. *Suppose* $F(T) = \sum_{i=0}^{\infty} a_i T^i \in L[[T]]$. *Let* $A_{s\,m}$ *be the* $m+1 \times m+1$ *matrix* $(b_{i\,j})_{0 \leq i,j \leq m}$, *where* $b_{i\,j} = a_{s+i+j}$. *Then* $F(T)$ *is the expansion at 0 of a rational function if and only if there exists* $m \geq 0$ *such that* $\det A_{s\,m} = 0$ *for* s *sufficiently large.*

Now suppose $m \geq 0$ is such that $\det A_{s\,m} = 0$ for $s \geq S$ and there is no smaller m with this property. Claim: $\det A_{s\,m-1} \neq 0$ for $s \geq S$.

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{pmatrix}$$

Lemma. $Z(X,T) \in \mathbf{Z}[[T]]$.

Lemma. *The coefficient of* T^i *in* $Z(X,T)$ *is at most* q^{ni}.

Theorem. $Z(X,T)$ *is meromorphic.*

Weierstrāss Preperation

For any R, $Z(X,T) = F_R(T)/P_R(T)$.