Algebraic Number Theory

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Lecture 25

A rationality criterion

Lemma. *Suppose* $F(T) = \sum_{i=0}^{\infty} a_i T^i \in L[[T]]$. *Let* $A_{s\,m}$ *be the* $m+1 \times m+1$ *matrix* $(b_{i\,j})_{0 \le i,j \le m}$, *where* $b_{i\,j} = a_{s+i+j}$. *Then* $F(T)$ *is the expansion at 0 of a rational function if and only if there exists* $m \ge 0$ *such that* $\det A_{s\,m} = 0$ *for* s *sufficiently large. Proof. Suppose*

$$F(T) = \frac{P(T)}{Q(T)}$$

where $P(T)$ *is a polynomial of degree* M *ancd*

$$Q(T) = \sum_{i=0}^{N} c_i T^i.$$

Then

$$a_n c_0 + a_{n-1} c_1 + \cdots + a_{n-N} c_N = 0$$

if $n > M$. *Thus* $\det A_{n-N,N} = 0$ *for* $s > M - N$.

Now suppose $m \ge 0$ is such that $\det A_{s\,m} = 0$ for $s \ge S$ and there is no smaller m with this property. Claim: $\det A_{s\,m-1} \ne 0$ for $s \ge S$.

Suppose the claim for now. Then for $s \ge S$ some non-trivial linear combination of the rows of $A_{s\,m}$ vanishes but the coefficient of the last row can't be zero.

Thus for $s \ge M$ $(a_{s+m}, \ldots, a_{s+2m})$ is a linear combination of

$$(a_s, \ldots, a_{s+m})$$

$$\cdots \qquad \cdots$$

$$(a_{s+m-1}, \ldots, a_{s+2m-1})$$

It follows that if $u_0, u_1, \ldots, u_m$ are such that

$$a_S u_m + \ldots + a_{S+m} u_0 = 0$$

$$\ldots \qquad \ldots$$

$$a_{S+m-1} u_m + \ldots + a_{S+2m-1} u_0 = 0$$

then

$$a_s u_m + \ldots + a_{s+m} u_0 = 0$$

for $s \geq S$. This implies

$$(\sum_{i=0}^{m} u_i T^i) \cdot F(T)$$

is a polynomial.