Algebraic Number Theory

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Lecture 19

The Zeta function and cc operators

Lemma. *Ψ_q is completely continuous.*

Proof. Recall

$$\Psi_q\colon \sum_{w\in\mathbf{N}^n} b_w X^w \mapsto \sum_{w\in\mathbf{N}^n} b_{qw} X^w.$$

If $F \in S_{n,M}$, $\Psi_q(F) \in S_{n,M}$.

Proposition. *If L is CC, $\det(1 - \pi_m \circ L|_{S_m} T)$ converges to an entire series $\det(1 - LT)$.*

Let $-\operatorname{Tr}L$ be the linear coefficient of $\det(1 - LT)$.

Lemma.

$$\det(1 - LT) = \exp(-\sum_{n=1}^{\infty} \frac{TrA^n}{n} T^n).$$

Proof.

Lemma. *If $G \in S_{n,M}$.*

$$(q^s - 1)^n \ Tr(\Psi_q \circ m_G)^s = \sum_{\substack{x\in\mathbf{C}_p^n \\ x^{q^s-1}=1}} G(x)G(x^q)\cdots G(x^{q^{s-1}})$$

Recall, if $f \in k[x_1, \ldots, x_r]$ and $X(R) = \{s \in R^n\colon f(s) = 0\}$. The zeta function of X is

$$Z(X, T) = \exp(\sum_{n\geq 1} \frac{|X(k_n)|}{n} T^n.$$

Let $X'(R) = \{s \in X(R) \colon x_i(s) \in R^*\}$ and

$$Z(X', T) = \exp(\sum_{n \geq 1} \frac{|X'(k_n)|}{n} T^n.$$